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The force–moments on a deforming body introduced impulsively into an inviscid incompressible fluid with uniform vorticity

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Abstract

This paper is concerned with the derivation of dynamical equations for freely deforming bodies with more than six degrees of freedom which are immersed in an inviscid incompressible fluid. Following Proudman's pioneering work for a sphere our method is applied to a fluid with uniform vorticity but otherwise arbitrary non-uniform strain-rate at the instant after the body has been impulsively introduced into the fluid. The rotational disturbance field is consequently zero thus enabling the generalised force–moments of arbitrary order to be determined from a Laplace problem through the use of Green's theorem and generalised Kirchhoff potentials. An infinite system of equations is obtained each which contains an inertial term, given by the rate of change of the generalised Kelvin Impulse, a generalised lift, a deformation-induced surface momentum flux and a surface kinetic energy. The assumption of an impulsive start places no constraint on the use of our force–moment formulae in irrotational flow but they can only be applied at the starting instant in rotational flow or, when the strain-rate is weak, for early times in the body's motion. Nonetheless, the start conditions for the rotational case can be created experimentally and be applied to initially free tumbling bodies when they start to deform. This newly identified equation system provides the foundation for new analytical and numerical approaches to the macroscopic modelling of freely deforming bodies and bubbly two-phase flow. In particular, the equations show that the added masses are not sufficient to characterise the body's geometry and that independent geometric constants are also required, here referred to as the added Kirchhoff energies. Finally, the zero- and first-order force–moment equations are used to derive the force and torque that apply to bodies with six degrees of freedom and their analytic forms are shown to agree with independent results for arbitrarily shaped deforming bodies in both rotational and irrotational flows.

Keywords: Deforming; Body; Bubble; Force; Dynamics

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1. Introduction

The dynamics of bodies with the six degrees of freedom, as given by their linear and angular velocities, are completely determined by the applied force and torque. Consequently, only three force equations and three torque equations are required for solid bodies or wholly self propelled bodies (Miloh and Galper, 1993). More equations are required for bodies whose surfaces are free to deform. Bubbles, for example, have infinitely many degrees of freedom. Self-propelled bodies such as a kite or a jelly fish are partly free to deform since, in addition to the tow rope or muscle-driven body which powers them forwards, they also have free steamers that stabilise their motion. This paper principally concerns deforming bodies with more than six degrees of freedom. We shall derive equations for the force–moment of arbitrary order. The force–moment of order n, here denoted $\mathbf{m}^{(n)}$, is defined in terms of the fluid pressure p, the normal $\mathbf{n}$ to the body surface $\mathscr{S}$ and the nth tensor power $\mathbf{x}^{(n)}$ of the position vector $\mathbf{x}$ relative to the centroid thus

$$\mathbf{m}^{(n)} = \int_{\mathscr{S}} -p\mathbf{n} \otimes \mathbf{x}^{(n)} \, \mathrm{d}\mathscr{s}, \tag{1.1a}$$

$$\mathbf{x}^{(0)} = 1, \quad \mathbf{x}^{(1)} = \mathbf{x}, \quad \mathbf{x}^{(2)} = \mathbf{x} \otimes \mathbf{x}, \ldots \tag{1.1b}$$

Here n is a non-negative integer and $\otimes$ denotes the tensor product with a vector. The net force $\mathbf{f}$ then equals our zero-order force moment $\mathbf{m}^{(0)}$ and the torque $\mathbf{j}$ the double scalar product of the permutation tensor $\boldsymbol{\varepsilon}$ with our first-order force moment $\mathbf{m}^{(1)}$, namely $\mathbf{j} = \boldsymbol{\varepsilon} : \mathbf{m}^{(1)}$.

Our method assumes that the body is impulsively introduced into an ambient flow field with uniform vorticity but otherwise arbitrary spatial velocity gradients. This approach renders the rotational disturbance velocity zero thus enabling the application of the theory of the Laplace equation. The impulsive start constrains our results for rotational flows to early times which otherwise apply for all time in irrotational flows. It is important to note, as described more fully later in Section 1, that the conditions of our impulsive start are physically realised at the instant a free tumbling body starts to deform.

The literature concerning the force and torque on bodies, both rigid and deforming, in inviscid incompressible fluids is substantial. Approximating real fluids as inviscid is valid in high Reynolds number bubble flows (Legendre and Magnaudet, 1998; Magnaudet and Eames, 2000) and aero-acoustics (Howe, 1995). This is because the boundary conditions can be approximated as free-slip since any vorticity is confined to a narrow boundary layer. The special cases of irrotational and two dimensional rotational ambient flow are discussed in the classical literature (Lamb, 1945; Batchelor, 1967). Their generalisation to arbitrary body shapes and spatial gradients of the ambient flow field are documented thoroughly (Landweber and Miloh, 1980; Galper and Miloh, 1994, 1995). However, there still remain unsolved problems for the three dimensional rotational case (Miloh, 2003) principally because the existence of a non-zero rotational disturbance acceleration and/or velocity field complicates the mathematical analysis. The literature concerning the force and torque on three dimensional bodies in rotational flows is much smaller. Since our focus is on the impulsively started case we shall only make passing reference to the solutions that have been obtained for the sphere (Auton, 1987; Auton et al., 1988) and arbitrarily shaped body (Catlin, 2003) when immersed in a steady linear shear flow. These analyses differ from our interest here by involving a fully developed and steady rotational disturbance field.

Since our methods are closely related it is instructive to discuss Proudman's (1916) pioneering work concerning the force on a sphere located centrally on the axis of a rotating column of fluid. Proudman formulates his problem in the fluid-fixed rotating reference frame in which the flow is initially irrotational. Since the fluid can rotate freely around the sphere there is no initial disturbance field. In his (47) Proudman applies Green's theorem to determine his generalised lift force Q_r'' at the instant a linear velocity is imparted to the sphere. Note that his total force is obtained by adding to Q_r'' both a centrifugal term Q_r''', his (48), and a potential term Q_r' the latter which, since it involves the time derivative of his velocity potential ϕ, takes different values in the laboratory and rotating frames. As a consequence, his generalised lift Q_r'' also takes different values in the two reference frames.

Importantly Proudman's analysis led to the discovery of the Taylor column in bounded flows (Greenspan, 1968) which was subsequently demonstrated experimentally by Taylor (1921). Strictly, the complicated question of whether or not a Taylor column forms does not concern us here since our analysis for a rotational ambient flow applies at the starting instant. However, as Miloh (2003) argues in his Section 2, our analysis also applies at early times in the body's motion provided that the ambient strain-rate is sufficiently weak. Taylor columns form in strong vorticity, or equivalently at low Rossby number (Greenspan, Section 1). When the strain-rate is weak the rotational disturbance field is then dominated by distortion of the ambient vorticity by the irrotational disturbance field (Lighthill, 1956, 1957b,c; Catlin, 2003) and consequently the complicated non-linear interactions that cause Taylor columns are not present. In particular the distortion due to the irrotational field, and consequently the rotational disturbance velocity, both increase linearly with time. This is the basis for being able to neglect at early times the rotational disturbance field in flows with weak strain-rate (Miloh, 2003, Section 2).

Recently Proudman's method has been applied again to the sphere in a rotating reference frame but assuming a more general ambient strain-rate (Drew and Lahey, 1987). In addition Drew and Passman (1999) claim to recover Proudman's lift force by conducting their analysis in the inertial laboratory reference frame. More recently two different approaches have also been applied to the impulsively introduced body. Legendre and Magnaudet (1998) determine the force on a sphere in an impulsively started linear shear flow. Miloh (2003) applies a method due to Quarterpelle and Napolitano (1982) to determine the force and torque on an arbitrarily shaped deforming body in an ambient flow with uniform vorticity. Miloh's method involves splitting the pressure field into singular and regular parts which are evaluated separately. He assumes that for early times the disturbance vorticity and, consequently, the rotational disturbance field can be neglected. Note that Miloh's analysis is the more general and his results, therefore, supersede the other references in this paragraph.

Our method, like Proudman's, will be to conduct our analysis at the instant the body is impulsively introduced into the flow. Since there is no time for the rotational disturbance vorticity to form, it being caused by the distortion of the ambient vortex tubes (Batchelor, Section 5.3), the disturbance field is necessarily zero. The assumption of impulsive introduction places no constraints on our results for irrotational flows since, being incompressible, the disturbance field forms instantaneously and, therefore, our results apply at any time in the body's motion. For a rotational ambient flow our neglect of the rotational disturbance velocity constrains the application of our results to the instant the body starts to deform or, when the strain-rate is weak, to early times.

Nonetheless, our assumed starting conditions can be readily created experimentally in a similar way to Taylor's (1921) classical experiment. The body can be rigidly suspended from a closed cylindrical fluid container, the latter which is rotated at fixed angular velocity about its axis. In time viscosity will damp out all disturbances resulting from the starting motion leaving the whole system, fluid, cylindrical container and body, rotating in a solid body motion. Note that it is not necessary for body's centroid to be coincident with the rotation axis and that the body can have arbitrary shape. The impulsive start can then be created in different ways. The body may just be allowed to deform. Alternatively, the rotation rate of the container can be suddenly changed with or without body deformation. Thus the initial flow conditions around the body in both these cases coincides exactly with the assumptions of our analysis. Study of the body force in such experiments may help elucidate the mechanisms that cause Taylor columns.

In Section 2 we shall formulate the problem and define our generalised Kelvin Impulse and generalised added masses. We shall work with disturbance forces which correspond to the interphase forces (Kowe et al., 1988, Section 3) that arise in the formulation of two-phase flow equations. From a mathematical perspective, this approach avoids the regular terms, namely those with no singularities in the interior of the body. In Section 3 we formulate a Laplace problem and derive the force–moments by applying Green's theorem using our generalised Kirchhoff potentials as Green's functions. In Section 4 we express the generalised force–moments in terms of the spatial gradients of the ambient velocity field, the generalised added masses and our newly identified generalised added Kirchhoff energies. The conventional force and torque equations are derived from our generalised force–moment equations in Section 5. In Section 6 we compare our results with the independent work of Miloh (2003) and Galper and Miloh (1994) for the force and torque in both irrotational and rotational ambient flows. Finally in Section 7 we discuss the application of our general equation set to determining the dynamics of freely deforming bodies.

2. Problem formulation

2.1. Field variables and momentum equations

Regarding our notation we shall use the superscript n enclosed in brackets to denote the nth power and so for tensor products we have $\mathbf{x}^{(0)} = 1$, $\mathbf{x}^{(1)} = \mathbf{x}$, $\mathbf{x}^{(2)} = \mathbf{x} \otimes \mathbf{x}$, hence $\mathbf{x}^{(n)}$ and similarly $\cdot^{(n)}$ for the nth-order scalar product. Similarly for the grad operator $\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ we define $\boldsymbol{\nabla}^{(2)} = \boldsymbol{\nabla} \otimes \boldsymbol{\nabla}$. In particular the strain-rate is given by $(\mathbf{U} \otimes \boldsymbol{\nabla})_{ij} = \mathbf{U}_{i,j} = (\boldsymbol{\nabla} \otimes \mathbf{U})^{\mathrm{T}}$ and the vector-product by $\mathbf{a} \wedge \mathbf{b} = \boldsymbol{\varepsilon} : (\mathbf{b} \otimes \mathbf{a})$ where colon represents double scalar summation (two orders reducing). In general we shall interpret scalar products as summation over adjacent indices unless indicated otherwise by index notation.

Our analysis, like that of Auton et al. (1988, Section 2.2), is conducted in the non-inertial reference frame $\mathbf{x}$ that moves parallel to the inertial laboratory frame $\tilde{\mathbf{x}}$ and whose origin is

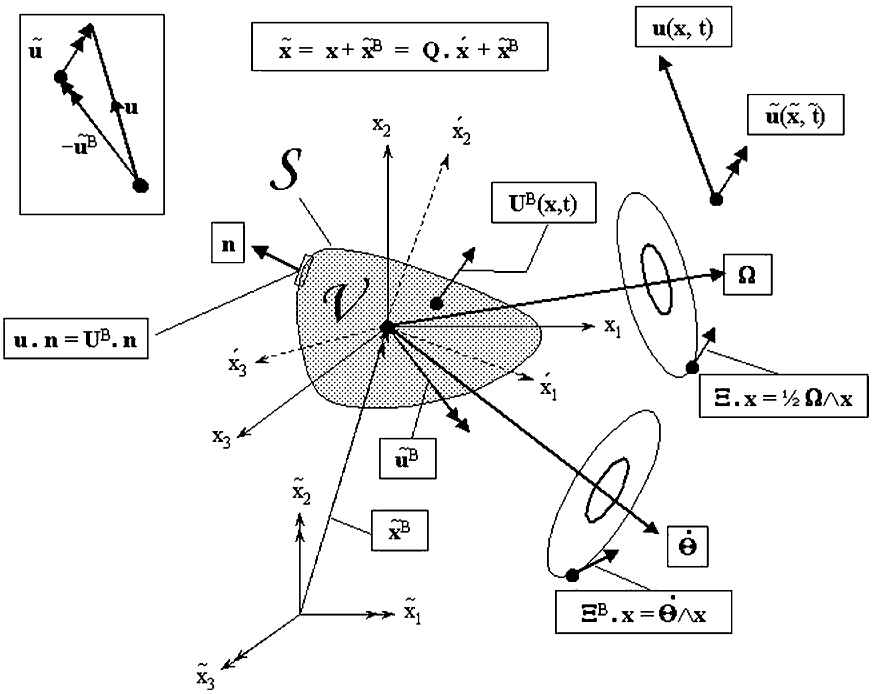

Fig. 1. Coordinate systems, fluid and body velocities: Absolute (solid double arrow); relative (solid single arrow); rotating (dashed single arrow); angular (circled arrow).

coincident with the body's centroid $\tilde{\mathbf{x}}^{\mathrm{B}}$ (Fig. 1). Our reference frame is not in general inertial since we allow the body to accelerate, namely $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\mathbf{x}}^{\mathrm{B}} \neq \mathbf{0}$. The ambient *relative* fluid velocity $\mathbf{U}$ in the non-inertial reference frame is assumed to have both an unsteady translation $\mathbf{U}|_{\mathbf{0}}$ and an unsteady spatially non-uniform strain-rate $(\nabla \otimes \mathbf{U})^{\mathrm{T}}$. Here $|_{\mathbf{0}}$ is used to denote evaluation of $\mathbf{U}(\mathbf{x}, t)$ at the centroid and coordinate origin $\mathbf{x} = \mathbf{0}$, namely $\mathbf{U}|_{\mathbf{0}} = \mathbf{U}(\mathbf{0}, t)$. Thus if we denote the *absolute* fluid velocity in the inertial laboratory frame as $\widetilde{\mathbf{U}}$, and the centroid velocity by $\tilde{\mathbf{u}}^{\mathrm{B}}(= \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{x}}^{\mathrm{B}})$, our relative fluid velocity $\mathbf{U}|_{\mathbf{0}}$ is equal to the difference of the absolute velocities in the laboratory frame thus $\mathbf{U}|_{\mathbf{0}} = \widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}} - \tilde{\mathbf{u}}^{\mathrm{B}}$. Here $|_{\tilde{\mathbf{x}}^{\mathrm{B}}}$ denotes evaluation of the absolute velocity field $\widetilde{\mathbf{U}}(\tilde{\mathbf{x}}, \tilde{\mathbf{t}})$ at the centroid $\tilde{\mathbf{x}}^{\mathrm{B}}$, namely $\widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}} = \widetilde{\mathbf{U}}(\tilde{\mathbf{x}}^{\mathrm{B}}, \tilde{\mathbf{t}})$. The spatial gradients $\nabla^{(n)} \otimes \mathbf{U}$ of the ambient fluid velocity, therefore, are the same in both reference frames, namely $\nabla^{(n)} \otimes \mathbf{U} = \widetilde{\nabla}^{(n)} \otimes \widetilde{\mathbf{U}}$ for $n > 0$.

The strain-rate $(\nabla \otimes \mathbf{U})^{\mathrm{T}}$ (Batchelor, 1967, Section 2.3) has both symmetric $\mathbf{E}$ and anti-symmetric $\boldsymbol{\Xi}$ parts, the former determining the irrotational part and the latter, which is related to the vorticity $\boldsymbol{\Omega}$, determining the rotational part of the strain-rate induced fluid velocity. In our tensor product notation then

$$\mathbf{E} = \tfrac{1}{2}[(\mathbf{U} \otimes \nabla) + (\nabla \otimes \mathbf{U})], \quad \boldsymbol{\Xi} = \tfrac{1}{2}[(\mathbf{U} \otimes \nabla) - (\nabla \otimes \mathbf{U})], \quad \boldsymbol{\Xi} = -\tfrac{1}{2}\boldsymbol{\varepsilon} \cdot \boldsymbol{\Omega}. \tag{2.1}$$

Again note that the above relative ambient strain-rates and vorticity have the same values in the inertial laboratory reference frame, namely $\mathbf{E} = \widetilde{\mathbf{E}}$, $\boldsymbol{\Xi} = \widetilde{\boldsymbol{\Xi}}$ and $\boldsymbol{\Omega} = \widetilde{\boldsymbol{\Omega}}$.

While we shall be restricted to a spatially uniform ambient vorticity our analysis is otherwise applicable to an arbitrary irrotational field. However, for our analysis to apply for early times in the body's deformation the ambient vorticity field must remain uniform and constant. To explore the conditions under which this holds true consider the ambient vorticity transport equation (Batchelor, 1967, Eq. (5.1.2), p. 267)

$$\frac{\partial}{\partial t}\boldsymbol{\Omega} + \mathbf{U}\cdot(\boldsymbol{\nabla}\otimes\boldsymbol{\Omega}) = \boldsymbol{\Omega}\cdot(\boldsymbol{\nabla}\otimes\mathbf{U}). \tag{2.2a}$$

Since we are assuming that the vorticity field is initially spatially uniform then $\boldsymbol{\nabla}\otimes\boldsymbol{\Omega} = \mathbf{0}$ and further, since by definition $\boldsymbol{\Omega}\cdot(\boldsymbol{\nabla}\otimes\mathbf{U}) = (\boldsymbol{\nabla}\otimes\mathbf{U})^{\mathrm{T}}\cdot\boldsymbol{\Omega} = (\mathbf{E}+\boldsymbol{\Xi})\cdot\boldsymbol{\Omega}$ and $\boldsymbol{\Xi}\cdot\boldsymbol{\Omega} = -\frac{1}{2}\boldsymbol{\Omega}\wedge\boldsymbol{\Omega} = \mathbf{0}$, we find from (2.2a) that the rate of change of the ambient vorticity is given by

$$\frac{\partial}{\partial t}\boldsymbol{\Omega} = \mathbf{E}\cdot\boldsymbol{\Omega}. \tag{2.2b}$$

Thus our requirements on the ambient vorticity are met provided both the irrotational and rotational parts of the strain-rate are weak so that their product can be neglected and the vorticity then remains uniform and constant.

We shall adopt the general convention of using lower-case letters to represent total fluid velocity field variables, capitals to represent the ambient (regular) field variables and the prefix Δ to denote the disturbance field variables. The ambient relative velocity field $\mathbf{U}$ is split into irrotational $\mathbf{V}$ and rotational $\mathbf{W}$ components, the latter with spatially uniform vorticity $\boldsymbol{\Omega}$ and the former with velocity potential Φ. The total field variables are then expressible in terms of the ambient field and the disturbance field thus

$$\mathbf{u} = \mathbf{v} + \mathbf{w}, \quad \mathbf{u} = \mathbf{U} + \boldsymbol{\Delta}\mathbf{u}, \tag{2.3a}$$

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\Delta}\mathbf{v} = \boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\Delta\varphi, \quad \mathbf{w} = \mathbf{W} + \boldsymbol{\Delta}\mathbf{w}, \tag{2.3b}$$

$$\varphi = \Phi + \Delta\varphi, \quad \boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\Delta\omega}, \tag{2.3c}$$

$$p = \mathbf{P} + \Delta p. \tag{2.3d}$$

Similarly $\mathbf{v}$ and $\mathbf{w}$ are the irrotational and rotational components of the total velocity field their uniqueness being determined by the requirement that $\boldsymbol{\Delta}\mathbf{w}$ has zero surface flux, namely $\boldsymbol{\Delta}\mathbf{w}\cdot\mathbf{n}|_{\mathscr{S}} = 0$. The disturbance pressure Δp is defined as the difference between the total pressure p and its ambient value P. This definition of the disturbance field then ensures that all the disturbance quantities are independent of the reference frame, namely $\Delta\varphi = \Delta\tilde{\varphi}$, $\boldsymbol{\Delta}\mathbf{v} = \boldsymbol{\Delta}\tilde{\mathbf{v}}$, $\boldsymbol{\Delta\omega} = \boldsymbol{\Delta}\tilde{\boldsymbol{\omega}}$ and $\boldsymbol{\Delta}\mathbf{w} = \boldsymbol{\Delta}\tilde{\mathbf{w}}$.

By writing the advection term $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}$ in terms of the vorticity thus, $\mathbf{u}\cdot(\boldsymbol{\nabla}\otimes\mathbf{u}) = \frac{1}{2}\boldsymbol{\nabla}\otimes(\mathbf{u}\cdot\mathbf{u}) + \boldsymbol{\omega}\wedge\mathbf{u}$, and the irrotational velocity $\mathbf{v}$ in terms of the velocity potential as $\mathbf{v} = \boldsymbol{\nabla}\varphi$ the momentum equations for the total and ambient relative fields become

$$\frac{\partial}{\partial t}\mathbf{w} + \boldsymbol{\nabla}\otimes\left\{\frac{\partial}{\partial t}\varphi + \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{1}{\rho^0}p + \mathbf{x}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{u}}^{\mathrm{B}}\right\} + \boldsymbol{\omega}\wedge\mathbf{u} = \mathbf{0} \tag{2.4a}$$

and

$$\frac{\partial}{\partial t}\mathbf{W} + \boldsymbol{\nabla}\otimes\left\{\frac{\partial}{\partial t}\Phi + \frac{1}{2}\mathbf{U}\cdot\mathbf{U} + \frac{1}{\rho^0}P + \mathbf{x}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{u}}^{\mathrm{B}}\right\} + \boldsymbol{\Omega}\wedge\mathbf{U} = \mathbf{0}. \tag{2.4b}$$

Here $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{u}}^{\mathrm{B}}(=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\mathbf{x}}^{\mathrm{B}})$ is the acceleration of the centroid with respect to the laboratory reference frame.

2.2. Boundary conditions and the Kirchhoff potential expansion

As explained in the introduction we are only concerned with cases in which both the disturbance vorticity and, necessarily, the rotational disturbance velocity are identically zero, namely

$$\boldsymbol{\Delta\omega} = \mathbf{0}, \quad \boldsymbol{\Delta}\mathbf{w} = \mathbf{0}. \tag{2.5}$$

The irrotational disturbance velocity $\boldsymbol{\Delta}\mathbf{v}$, therefore, satisfies the following boundary condition on the surface of the body

$$\mathbf{u} \cdot \mathbf{n}|_{\mathscr{S}} = (\mathbf{U} + \boldsymbol{\Delta}\mathbf{v}) \cdot \mathbf{n}|_{\mathscr{S}} = \mathbf{U}^{\mathrm{B}} \cdot \mathbf{n}|_{\mathscr{S}}, \tag{2.6a}$$

or equivalently

$$\boldsymbol{\Delta}\mathbf{v} \cdot \mathbf{n}|_{\mathscr{S}} = -\boldsymbol{\delta}\mathbf{U} \cdot \mathbf{n}|_{\mathscr{S}}, \tag{2.6b}$$

where $\boldsymbol{\delta}\mathbf{U}$ is the difference velocity between the ambient relative fluid velocity $\mathbf{U}$ and the relative continuum velocity of the body $\mathbf{U}^{\mathrm{B}}$. Here $\mathbf{U}^{\mathrm{B}}$ is the regular function describing the velocity of the body's continuum relative to the centroid. In practice we are only interested in the surface velocity of the body but to aid our mathematical analysis we shall regard $\mathbf{U}^{\mathrm{B}}$ as the analytic continuation of the body's surface velocity into its interior. The difference velocity $\boldsymbol{\delta}\mathbf{U}$, therefore, has the following Taylor expansion:

$$\boldsymbol{\delta}\mathbf{U} = \mathbf{U} - \mathbf{U}^{\mathrm{B}} = \sum_{m=0}^{\infty} \frac{1}{\mathbf{m!}} \mathbf{x}^{(m)} \cdot^{(m)} (\boldsymbol{\nabla}^{(m)} \otimes \boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}. \tag{2.7}$$

The irrotational disturbance velocity boundary condition (2.6) then becomes

$$\begin{aligned}\boldsymbol{\Delta}\mathbf{v} \cdot \mathbf{n}|_{\mathscr{S}} &= -\sum_{m=0}^{\infty} \frac{1}{m!} \mathbf{n} \cdot [\mathbf{x}^{(m)} \cdot^{(m)} (\boldsymbol{\nabla}^{(m)} \otimes \boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}]|_{\mathscr{S}} \\ &= -\sum_{m=0}^{\infty} \frac{1}{m!} (\mathbf{n} \otimes \mathbf{x}^{(m)})|_{\mathscr{S}} \cdot^{(m+1)} (\boldsymbol{\nabla}^{(m)} \otimes \boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}\end{aligned} \tag{2.8}$$

and the corresponding disturbance velocity potential $\Delta\varphi$, therefore, has the following expansion in terms of our generalised Kirchhoff potentials $\boldsymbol{\psi}^{(m)}$

$$\Delta\varphi = -\sum_{m=0}^{\infty} \frac{1}{m!} \boldsymbol{\psi}^{(m)} \cdot^{(m+1)} (\boldsymbol{\nabla}^{(m)} \otimes \boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}} \tag{2.9a}$$

where the $\boldsymbol{\psi}^{(m)}$ are defined by their normal boundary conditions thus

$$\frac{\partial}{\partial n} \boldsymbol{\psi}^{(m)}|_{S} = (\mathbf{n} \otimes \mathbf{x}^{(m)})|_{\mathscr{S}}. \tag{2.9b}$$

Our $\boldsymbol{\psi}^{(m)}$ are a natural generalisation of the classical Kirchhoff potentials (Lamb, 1945, Chapter 6). Although the boundary condition $\boldsymbol{\Delta}\mathbf{w} \cdot \mathbf{n}|_{\mathscr{S}} = 0$ is satisfied trivially, since our rotational disturbance velocity is identically zero, it also enforces the following constraint on the rate of change of the normal flux:

$$\left(\frac{\partial}{\partial t}\mathbf{\Delta w}\right)\cdot\mathbf{n}|_{\mathscr{S}}=\mathbf{0}. \tag{2.10}$$

This follows from our normal flux requirement $\mathbf{\Delta w}\cdot\mathbf{n}|_{\mathscr{S}}=0$ by partially differentiating with respect to time thus $\frac{\partial}{\partial t}(\mathbf{\Delta w}\cdot\mathbf{n})=\left(\frac{\partial}{\partial t}\mathbf{\Delta w}\right)\cdot\mathbf{n}+\mathbf{\Delta w}\cdot\frac{\partial}{\partial t}\mathbf{n}$, and noting that the second right-hand term vanishes because $\mathbf{\Delta w}=\mathbf{0}$.

It is important to appreciate that the body's continuum velocity is not entirely arbitrary since our analysis requires that the centroid stay at the origin and $\mathbf{U}^{\mathrm{B}}$ must then satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathscr{V}}\mathbf{x}\,\mathrm{d}v\right]=\int_{\mathscr{S}}(\mathbf{U}^{\mathrm{B}}\cdot\mathbf{n})\mathbf{x}\,\mathrm{d}s=\mathbf{0}, \tag{2.11a}$$

which by applying the divergence theorem can also be expressed as

$$\sum_{m=0}^{\infty}\frac{1}{m!}(\mathbf{\nabla}^{(m)}\otimes\mathbf{U}^{\mathrm{B}})|_{\mathbf{0}}\cdot^{(m+1)}\int_{\mathscr{V}}\mathbf{\nabla}\otimes\mathbf{x}^{(m+1)}\,\mathrm{d}v=\mathbf{0} \tag{2.11b}$$

where the scalar summation should be interpreted as $\{(\mathbf{\nabla}^{(m)}\otimes U_j^{\mathrm{B}})|_{\mathbf{0}}\cdot^{(m)}\nabla_j\otimes(\mathbf{x}\otimes\mathbf{x}^{(m)})\}$. Thus (2.11b) defines a relationship between the gradients of the body's velocity and the body's generalised moments of volume $\mathscr{V}^{(m)}$ where

$$\mathscr{V}^{(m)}=\int_{\mathscr{V}}\mathbf{x}^{(m)}\,\mathrm{d}v. \tag{2.12}$$

2.3. The generalised added masses and their rates of change

We shall first define our generalised Kelvin Impulse like (3.6) and (4.33) of Galper and Miloh (1994) as

$$\mathscr{m}^{(n)}=\int_{\mathscr{S}}\mathbf{n}\otimes\mathbf{x}^{(n)}\Delta\varphi\,\mathrm{d}s=\int_{\mathscr{S}}\frac{\partial}{\partial n}\boldsymbol{\psi}^{(n)}\Delta\varphi\,\mathrm{d}s. \tag{2.13}$$

Again following (3.8)–(3.11) of Galper and Miloh the Kelvin Impulse admits the following expansion:

$$\mathscr{m}^{(n)}=\sum_{m=0}^{\infty}\frac{1}{m!}\mathscr{m}^{(n,m)}\cdot^{(m+1)}(\mathbf{\nabla}^{(m)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}} \tag{2.14a}$$

in terms of our generalised added masses $\mathscr{m}^{(n,m)}$, which by Green's theorem, must satisfy

$$\mathscr{m}^{(n,m)}=\int_{\mathscr{S}}-\frac{\partial}{\partial n}\boldsymbol{\psi}^{(n)}\otimes\boldsymbol{\psi}^{(m)}\,\mathrm{d}s=\int_{\mathscr{S}}-\boldsymbol{\psi}^{(n)}\otimes\frac{\partial}{\partial n}\boldsymbol{\psi}^{(m)}\,\mathrm{d}s. \tag{2.14b}$$

Note that the usual 6×6 added mass tensor **B** (Miloh, 2003, Appendix A; Galper and Miloh, 1994, Section 3; Lamb, 1945, Chapter 6) and its sub-matrices **T**, **Z**, **R** can be expressed in terms of our generalised added masses $\mathscr{m}^{(n,m)}$ as shown in our Appendix A.

As it is relevant to our later determination of the inertial forces, note that the time derivative of the generalised Kelvin Impulse is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{m}^{(n)} = \sum_{m=0}^{\infty}\frac{1}{m!}\left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{m}^{(n,m)}\ .^{(m+1)}\ (\nabla^{(m)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}} + \mathcal{m}^{(m,n)}\ .^{(m+1)}\ \frac{\mathrm{d}}{\mathrm{d}t}(\nabla^{(m)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}\right], \tag{2.15a}$$

where the time derivative of the spatial gradients of the difference velocity $\frac{\mathrm{d}}{\mathrm{d}t}(\nabla^{(m)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}$ are related to the absolute ambient velocity $\widetilde{\mathbf{U}}$, taking account of the centroid velocity, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\nabla^{(m)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}\right] = \left(\widetilde{\nabla}^{(m)}\otimes\left[\frac{\partial}{\partial\tilde{t}}\widetilde{\mathbf{U}}\right]\right)|_{\tilde{\mathbf{x}}^{\mathrm{B}}} + \tilde{\mathbf{u}}^{\mathrm{B}}\cdot(\widetilde{\nabla}^{(m+1)}\otimes\widetilde{\mathbf{U}})|_{\tilde{\mathbf{x}}^{\mathrm{B}}} - \frac{\mathrm{d}}{\mathrm{d}t}[(\nabla^{(m)}\otimes\mathbf{U}^{\mathrm{B}})|_{\mathbf{0}}]. \tag{2.15b}$$

In general, our added masses $\mathcal{m}^{(n,m)}$ have non-zero time derivatives because their boundary conditions (2.9b) are time dependent. However, for rigid bodies their time derivative can be removed from the equations by expressing $\mathcal{m}^{(n,m)}$ in terms of its value $\acute{\mathcal{m}}^{(n,m)}$ in the rotating frame where it is constant. Since $\mathcal{m}^{(n,m)}$ is a $(m+n+2)$th-order tensor then, following Aris (1962, Section 2), it transforms to the rotating frame as

$$\mathcal{m}^{(n,m)} = \mathbf{Q}^{(n+m+2)}\ .^{(n+m+2)}\ \acute{\mathcal{m}}^{(n,m)} \tag{2.16}$$

where $\mathbf{Q}$ is the body's rotation tensor. The scalar product $\mathbf{Q}^{(n+m+2)}.^{(n+m+2)}$ should in the particular case $n=m=0$ be interpreted as $\mathcal{m}^{(0,0)} = \mathbf{Q}^{(2)} : \acute{\mathcal{m}}^{(0,0)} = \mathrm{Q}_{j\beta}\mathrm{Q}_{i\alpha}\acute{\mathcal{m}}^{(0,0)}_{\alpha\beta}$. The evaluation of the time derivative of products of the rotation tensor is much simplified by assuming our body-centred coordinate system $\mathbf{x}$ is aligned to the body-fixed rotating coordinate system $\acute{\mathbf{x}}$ at the instant of the analysis and, therefore, $\mathrm{Q}_{jk}=\delta_{jk}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Q}_{jk} = -\varepsilon_{jk\ell}\dot{\Theta}_\ell$ where $\dot{\Theta}$ is the angular velocity of the body. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{m}^{(n,m)} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Q}^{(n+m+2)}\right]\ .^{(n+m+2)}\ \acute{\mathcal{m}}^{(n,m)} \tag{2.17a}$$

and in the case $n=m=0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{m}^{(0,0)} = \acute{\mathcal{m}}^{(0,0)}\cdot(\boldsymbol{\varepsilon}.\dot{\boldsymbol{\Theta}}) - (\boldsymbol{\varepsilon}\cdot\dot{\boldsymbol{\Theta}})\cdot\acute{\mathcal{m}}^{(0,0)}. \tag{2.17b}$$

3. The disturbance force–moments expressed as Green's integrals

We shall consider only the generalised disturbance force–moment $\boldsymbol{\Delta}\mathbf{m}^{(n)}$ on the body which is defined in terms of the disturbance pressure Δp by

$$\frac{1}{\rho^0}\boldsymbol{\Delta}\mathbf{m}^{(n)} = \int_S -\frac{1}{\rho^0}\Delta p\,\mathbf{n}\otimes\mathbf{x}^{(n)}\,\mathrm{d}\mathcal{s}. \tag{3.1a}$$

The ambient force–moment $\mathbf{M}^{(n)}$, namely that corresponding to the ambient fluid pressure P, is evaluated in Appendix B. The total force–moment $\mathbf{m}^{(n)}$ is then given by the sum of disturbance and ambient components by

$$\mathbf{m}^{(n)} = \boldsymbol{\Delta}\mathbf{m}^{(n)} + \mathbf{M}^{(n)}. \tag{3.1b}$$

First consider the momentum equation for the disturbance field as given by the difference of the total and ambient fluid momentum equations (2.4) as

$$\frac{\partial}{\partial t}\mathbf{\Delta w} + \mathbf{\nabla}\left\{\frac{\partial}{\partial t}\Delta\varphi + \frac{1}{2}(\mathbf{u}\cdot\mathbf{u} - \mathbf{U}\cdot\mathbf{U}) + \frac{1}{\rho^0}\Delta p\right\} + \mathbf{\Omega}\wedge\mathbf{\Delta u} + \mathbf{\Delta}\boldsymbol{\omega}\wedge\mathbf{u} = \mathbf{0}. \tag{3.2}$$

Under our assumption that the body in introduced impulsively into the fluid the disturbance vorticity $\mathbf{\Delta}\boldsymbol{\omega}$ and rotational disturbance velocity $\mathbf{\Delta w}$ are both zero and the disturbance velocity $\mathbf{\Delta u}$ is then equal to its irrotational part, namely $\mathbf{\Delta u} = \mathbf{\Delta v}$. It now follows from (3.2) that

$$\mathbf{\nabla}\Psi = \frac{\partial}{\partial t}\mathbf{\Delta w} + \mathbf{\Omega}\wedge\mathbf{\Delta v} \tag{3.3a}$$

where Ψ is the scalar function

$$\Psi = -\left\{\frac{\partial}{\partial t}\Delta\varphi + \frac{1}{2}(\mathbf{u}\cdot\mathbf{u} - \mathbf{U}\cdot\mathbf{U}) + \frac{1}{\rho^0}\Delta p\right\}. \tag{3.3b}$$

Following Proudman (1916), we now take the divergence of (3.3a). Note that because the fluid is incompressible then $\mathbf{\nabla}\cdot\left(\frac{\partial}{\partial t}\mathbf{\Delta w}\right) = 0$ and because the vorticity is spatially uniform then $\mathbf{\nabla}\otimes\mathbf{\Omega} = \mathbf{0}$. Furthermore, since the irrotational disturbance velocity $\mathbf{\Delta v}$ satisfies $\mathbf{\nabla}\otimes\mathbf{\Delta v} = (\mathbf{\nabla}\otimes\mathbf{\Delta v})^{\mathrm{T}}$ then $\mathbf{\nabla}\cdot(\mathbf{\Omega}\wedge\mathbf{\Delta v}) = \mathbf{\Omega}\cdot[\boldsymbol{\varepsilon} : (\mathbf{\nabla}\otimes\mathbf{\Delta v})] = \mathbf{0}$. Remembering from (2.10) that rotational disturbance surface flux satisfies $\left(\frac{\partial}{\partial t}\mathbf{\Delta w}\right).\ \mathbf{n}|_{\mathscr{S}} = 0$ we find that Ψ satisfies the following Laplace problem in the fluid region outside the body.

$$\mathbf{\nabla}^2\Psi = 0, \quad \mathbf{\nabla}\Psi\cdot\mathbf{n}|_{\mathscr{S}} = (\mathbf{\Omega}\wedge\mathbf{\Delta v})\cdot\mathbf{n}|_{\mathscr{S}}. \tag{3.4}$$

When combined with the physical requirement that $\Psi\to 0$ as $|\mathbf{x}|\to+\infty$, Ψ being a function only of disturbance quantities, (3.4) defines a well posed Laplace problem for Ψ. Again following Proudman, now apply Green's theorem to Ψ and our generalised Kirchhoff potential $\boldsymbol{\psi}^{(n)}$, as defined by (2.9b), to give

$$\int_{\mathscr{S}}\Psi\mathbf{n}\otimes\mathbf{x}^{(n)}\,\mathrm{d}s = \int_{\mathscr{S}}\boldsymbol{\psi}^{(n)}(\mathbf{\Omega}\wedge\mathbf{\Delta v})\cdot\mathbf{n}\,\mathrm{d}s. \tag{3.5}$$

We can now recover an identity for the disturbance force–moment by substituting for Ψ from (3.3b) into (3.5), taking all terms except the pressure to the right-hand side, to give

$$\frac{1}{\rho^0}\mathbf{\Delta m}^{(n)} = \frac{1}{\rho^0}\mathbf{\Delta}\mathfrak{m}^{(n)} + \mathscr{g}^{(n)}, \tag{3.6a}$$

where

$$\frac{1}{\rho^0}\mathbf{\Delta}\mathfrak{m}^{(n)} = \int_{\mathscr{S}}\left\{\frac{\partial}{\partial t}\Delta\varphi + \frac{1}{2}(\mathbf{u}\cdot\mathbf{u} - \mathbf{U}\cdot\mathbf{U})\right\}\mathbf{n}\otimes\mathbf{x}^{(n)}\,\mathrm{d}s \tag{3.6b}$$

and

$$\mathscr{g}^{(n)} = \int_{\mathscr{S}}\boldsymbol{\psi}^{(n)}(\mathbf{\Omega}\wedge\mathbf{\Delta v})\cdot\mathbf{n}\,\mathrm{d}s = \int_{\mathscr{S}}\boldsymbol{\psi}^{(n)}\boldsymbol{\varepsilon} : (\mathbf{\Delta v}\otimes\mathbf{\Omega})\cdot\mathbf{n}\,\mathrm{d}s. \tag{3.6c}$$

Note that the Green's integral $\mathscr{g}^{(n)}$ only arises when the ambient field is rotational.

4. The disturbance force moments $\Delta \mathcal{M}^{(n)}$ in terms of added masses and energies

First we shall apply Leibnitz's differentiation theorem to the gradient of the nth moment of the disturbance velocity potential, namely the function $\nabla \otimes (\mathbf{x}^{(n)}\Delta\varphi)$, in the fluid region external to the body. There is no advective contribution from the surface at infinity because it is fixed relative to the non-inertial reference frame. We then obtain the rate of change of the generalised Kelvin Impulse expressed in terms of the partial time derivative of the disturbance velocity potential as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{m}^{(n)} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{S}} \Delta\varphi\,\mathbf{n}\otimes\mathbf{x}^{(n)}\,\mathrm{d}s\right] \\ &= \int_{\mathcal{S}}\left(\frac{\partial}{\partial t}\Delta\varphi\right)\mathbf{n}\otimes\mathbf{x}^{(n)}\,\mathrm{d}s + \int_{\mathcal{S}}\mathbf{u}\cdot\mathbf{n}\,\Delta\mathbf{v}\otimes\mathbf{x}^{(n)}\,\mathrm{d}s + \int_{S}(\nabla\otimes\mathbf{x}^{(n)})\Delta\varphi\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s. \end{aligned} \tag{4.1}$$

Now substituting (4.1) into our earlier identity (3.6) for the disturbance force–moment $\Delta\mathcal{M}^{(n)}$, whilst noting that on the surface of the body $\mathbf{u}\cdot\mathbf{n}|_{\mathcal{S}} = \mathbf{U}^{\mathrm{B}}\cdot\mathbf{n}|_{\mathcal{S}}$, then

$$\frac{1}{\rho^0}\Delta\mathcal{M}^{(n)} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{m}^{(n)} + \int_{\mathcal{S}}\left\{\frac{1}{2}(\mathbf{u}\cdot\mathbf{u}-\mathbf{U}\cdot\mathbf{U})\mathbf{n} - \mathbf{u}\cdot\mathbf{n}\,\Delta\mathbf{v}\right\}\otimes\mathbf{x}^{(n)}\,\mathrm{d}s - \int_{\mathcal{S}}(\nabla\otimes\mathbf{x}^{(n)})\Delta\varphi\,\mathbf{U}^{\mathrm{B}}\cdot\mathbf{n}\,\mathrm{d}s. \tag{4.2}$$

In the above equation the second right-hand integral represents the flux of the moment-of-momentum across the surface of the body caused by its deformation. The curly bracketed term in the first right-hand integral of (4.2) will now be transformed to allow it to be expressed in terms of added masses. Since the rotational disturbance velocity is zero ($\Delta\mathbf{w} = \mathbf{0}$) and the total velocity splits into the ambient velocity and irrotational disturbance velocity thus $\mathbf{u} = \mathbf{U} + \Delta\mathbf{v}$ we can then write

$$\frac{1}{2}(\mathbf{u}\cdot\mathbf{u}-\mathbf{U}\cdot\mathbf{U}) = \Delta\mathbf{v}\cdot\mathbf{U} + \frac{1}{2}\Delta\mathbf{v}\cdot\Delta\mathbf{v}. \tag{4.3}$$

When combined with the velocity boundary condition $\mathbf{u}\cdot\mathbf{n}|_{\mathcal{S}} = \mathbf{U}\cdot\mathbf{n}|_{\mathcal{S}} + \Delta\mathbf{v}\cdot\mathbf{n}|_{\mathcal{S}}$ then

$$\left\{\frac{1}{2}(\mathbf{u}\cdot\mathbf{u}-\mathbf{U}\cdot\mathbf{U})\mathbf{n} - \Delta\mathbf{v}(\mathbf{u}\cdot\mathbf{n})\right\}_{\mathcal{S}} = \langle\mathbf{n}\otimes\Delta\mathbf{v} - \Delta\mathbf{v}\otimes\mathbf{n}\rangle\cdot\mathbf{U}|_{\mathcal{S}} + \left(\frac{1}{2}\Delta\mathbf{v}\cdot\Delta\mathbf{v}\,\mathbf{I} - \Delta\mathbf{v}\otimes\Delta\mathbf{v}\right)\cdot\mathbf{n}|_{\mathcal{S}}, \tag{4.4}$$

where $\mathbf{I}(=\delta_{jk})$ is the second-order identity tensor. Equivalently

$$\left\{\frac{1}{2}(\mathbf{u}\cdot\mathbf{u}-\mathbf{U}\cdot\mathbf{U})\mathbf{n} - \mathbf{u}\cdot\mathbf{n}\,\Delta\mathbf{v}\right\}_{\mathcal{S}} = \langle\mathbf{n},\Delta\mathbf{v}\rangle\cdot\mathbf{U}|_{\mathcal{S}} + \mathbf{e}\cdot\mathbf{n}|_{\mathcal{S}}, \tag{4.5a}$$

where the anti-symmetric tensor commutator is given in full by

$$\langle\mathbf{n},\Delta\mathbf{v}\rangle = \langle\mathbf{n}\otimes\Delta\mathbf{v} - \Delta\mathbf{v}\otimes\mathbf{n}\rangle \tag{4.5b}$$

and $\mathbf{e}$ is our surface energy density tensor

$$\mathbf{e} = \frac{1}{2}\Delta\mathbf{v}\cdot\Delta\mathbf{v}\,\mathbf{I} - \Delta\mathbf{v}\otimes\Delta\mathbf{v}. \tag{4.5c}$$

Note that $\mathbf{e}$ is a symmetric tensor whose trace $\mathrm{Tr}\{\frac{1}{2}\Delta\mathbf{v}\cdot\Delta\mathbf{v}\,\mathbf{I} - \Delta\mathbf{v}\otimes\Delta\mathbf{v}\} = \frac{1}{2}\Delta\mathbf{v}\cdot\Delta\mathbf{v}$ is equal to the kinetic energy of the body's surface fluid.

Substituting our transformed identity (4.5) into (4.2) we find that the disturbance force–moment splits, respectively, into inertial, lift, surface momentum flux and surface kinetic energy components thus

$$\frac{1}{\rho^0}\Delta\mathsf{m}^{(n)} = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{m}^{(n)} + \mathscr{L}^{(n)} - \mathsf{D}^{(n)} - \mathscr{E}^{(n)}. \tag{4.6}$$

We have introduced negative signs in our deformation flux and surface energy terms to ensure their first moments $\mathsf{D}^{(1)}$ and $\mathscr{E}^{(1)}$ have positive definite values. Note that the first three right-hand terms in (4.6) can be expressed in terms of the generalised added masses. The inertial term $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{m}^{(n)}$ admits the expansion (2.14a). The surface momentum flux $\mathsf{D}^{(n)}$, whose integral form is

$$\mathsf{D}^{(n)} = \int_{\mathscr{S}} (\nabla\otimes\mathbf{x}^{(n)})\Delta\varphi\,\mathbf{U}^{\mathrm{B}}\cdot\mathbf{n}\,\mathrm{d}s, \tag{4.7}$$

has the more complicated expansion as derived in (D.1)–(D.3) of Appendix D. Importantly, because of the anti-symmetric properties of the tensor commutator, the generalised lift $\mathscr{L}^{(n)}$ can in general be transformed (Appendix C) thus

$$\mathscr{L}^{(n)} = \int_{\mathscr{S}} (\langle\mathbf{n},\Delta\mathbf{v}\rangle\cdot\mathbf{U})\otimes\mathbf{x}^{(n)}\,\mathrm{d}s = \int_{\mathscr{S}} \Delta\varphi\,\mathbf{n}\cdot\mathbf{\Pi} : [\nabla\otimes(\mathbf{U}\otimes\mathbf{x}^{(n)})]\,\mathrm{d}s \tag{4.8}$$

where $\mathbf{\Pi}$ is the fourth-order tensor $\mathbf{\Pi}_{qpk\ell} = (\delta_{q\ell}\delta_{pk} - \delta_{qk}\delta_{p\ell})$. Note that the right-hand integral above can be expressed as a linear combination of the generalised Kelvin Impulses which in turn can be expressed as added mass expansions. These expansions are derived explicitly in Appendix C for the zero- and first-order generalised lift.

In contrast, the surface energy $\mathscr{E}^{(n)}$, whose integral form is

$$\mathscr{E}^{(n)} = -\int_{\mathscr{S}} (\mathbf{e}\cdot\mathbf{n})\otimes\mathbf{x}^{(n)}\,\mathrm{d}s \tag{4.9}$$

cannot in general be expressed in terms of the added masses, but instead admits an expansion in terms of an independent class of geometric constants (Appendix D), which we shall refer to as added Kirchhoff energies $\mathscr{E}^{(n,m,p)}$, as given by (D.5b).

5. The force and torque for an ambient flow field with uniform strain-rate

To illustrate an application of the above generalised formulae we shall determine the force and torque when the ambient strain-rate of the fluid is uniform, namely when $(\nabla^{(m)}\otimes\mathbf{U})|_{\mathbf{0}} = \mathbf{0}$ for $m > 2$. We can, therefore, drop the evaluation at the centroid and write $(\nabla^{(1)}\otimes\mathbf{U})|_{\mathbf{0}} = \nabla\otimes\mathbf{U}$. Although we only need to determine the zero- and first-order force moments some of our following identities hold true for arbitrary order. First, by combining our (3.1), (3.6) and (4.6) the nth-order force moment $\mathbf{m}^{(n)}$ is given by

$$\frac{1}{\rho^0}\mathbf{m}^{(n)} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{m}^{(n)} + \mathscr{L}^{(n)} - \mathsf{D}^{(n)} - \mathscr{E}^{(n)} + \mathcal{g}^{(n)} + \frac{1}{\rho^0}\mathbf{M}^{(n)} \tag{5.1}$$

where $\mathbf{M}^{(n)}$ is the ambient or regular component. Further, with a uniform ambient strain-rate in the fluid it follows from (2.9) that the disturbance velocity potential takes the form

$$\Delta\varphi = -\boldsymbol{\psi}^{(0)} \cdot \mathbf{U}|_{\mathbf{0}} - \boldsymbol{\psi}^{(1)} : (\boldsymbol{\nabla} \otimes \mathbf{U}) + \sum_{m=1}^{\infty} \frac{1}{m!}\boldsymbol{\psi}^{(m)} \cdot^{(m+1)} (\boldsymbol{\nabla}^{(m)} \otimes \mathbf{U}^{\mathrm{B}})|_{\mathbf{0}}. \tag{5.2}$$

The first two right-hand terms in the above correspond to the disturbance field due to the presence of the body in the ambient flow and the infinite sum corresponds to the disturbance field caused by the body's deformation. Note that the summation starts at $m = 1$ since, by definition, the body's deformation velocity is zero at the centroid, namely $\mathbf{U}^{\mathrm{B}}|_{\mathbf{0}} = \mathbf{0}$. It now follows from (2.14a) that the Kelvin Impulse is given by

$$\mathcal{m}^{(n)} = \mathcal{m}^{(n,0)} \cdot \mathbf{U}|_{\mathbf{0}} + \mathcal{m}^{(n,1)} : (\boldsymbol{\nabla} \otimes \mathbf{U}) - \sum_{m=1}^{\infty} \frac{1}{m!}\mathcal{m}^{(n,m)} \cdot^{(m+1)} (\boldsymbol{\nabla}^{(m)} \otimes \mathbf{U}^{\mathrm{B}})|_{\mathbf{0}}. \tag{5.3}$$

The zero- and first-order generalised lift components are given by (C.10) as

$$\mathscr{L}^{(0)} = (\boldsymbol{\nabla} \otimes \mathbf{U})|_{\mathbf{0}} \cdot \mathcal{m}^{(0)} = \mathcal{m}^{(0)} \cdot (\boldsymbol{\nabla} \otimes \mathbf{U})^{\mathrm{T}}|_{\mathbf{0}} \tag{5.4a}$$

and

$$\mathscr{L}^{(1)} = (\mathcal{m}^{(0)} \cdot \mathbf{U}|_{\mathbf{0}}\mathbf{I} - \mathcal{m}^{(0)} \otimes \mathbf{U}|_{\mathbf{0}}) + [(\boldsymbol{\nabla} \otimes \mathbf{U})|_{\mathbf{0}}, \mathcal{m}^{(1)}] + (\boldsymbol{\nabla} \otimes \mathbf{U})|_{\mathbf{0}} : \mathcal{m}^{(1)}\,\mathbf{I}, \tag{5.4b}$$

where $\mathbf{I}$ is the identity tensor and $[\mathbf{A}, \mathbf{B}]$ $(= \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A})$ is the matrix commutator (Arnold and Khesin, 1998, Section 1.2). The first- and second-order surface momentum fluxes follow immediately from the integral form (4.7) as

$$\mathsf{D}^{(0)} = \mathbf{0}, \quad \mathsf{D}^{(1)} = \mathbf{I}\int_{\mathscr{S}} \Delta\varphi\,\mathbf{U}^{\mathrm{B}} \cdot \mathbf{n}\,\mathrm{d}\mathcal{s}. \tag{5.5}$$

The surface kinetic energy (4.9) can for our purpose be more conveniently expressed as a volume integral over the fluid region $V_\infty - V$ by applying the divergence theorem thus

$$\mathscr{E}^{(n)} = -\int_{\mathscr{S}} (\mathbf{e} \cdot \mathbf{n}) \otimes \mathbf{x}^{(n)}\,\mathrm{d}\mathcal{s} = -\int_{\mathscr{S}} \mathbf{n} \cdot (\mathbf{e} \otimes \mathbf{x}^{(n)})\,\mathrm{d}\mathcal{s} = \int_{\mathscr{V}_\infty - \mathscr{V}} \boldsymbol{\nabla} \cdot (\mathbf{e} \otimes \mathbf{x}^{(n)})\,\mathrm{d}\upsilon. \tag{5.6a}$$

By differentiating-out the far right integrand, noting that since the disturbance velocity $\boldsymbol{\Delta}\mathbf{v}$ is both incompressible and irrotational, then $\boldsymbol{\nabla} \cdot \mathbf{e} = \mathbf{0}$, $\boldsymbol{\nabla} \cdot \boldsymbol{\Delta}\mathbf{v} = \mathbf{0}$ and $\boldsymbol{\nabla} \otimes \boldsymbol{\Delta}\mathbf{v} = \boldsymbol{\Delta}\mathbf{v} \otimes \boldsymbol{\nabla}$ so we find

$$\mathscr{E}^{(n)} = \int_{\mathscr{V}_\infty - \mathscr{V}} \mathbf{e}^{\mathrm{T}} \cdot \boldsymbol{\nabla}(\mathbf{x}^{(n)})\,\mathrm{d}\upsilon. \tag{5.6b}$$

In particular for the zero- and first-order components

$$\mathscr{E}^{(0)} = \mathbf{0}, \quad \mathscr{E}^{(1)} = \int_{\mathscr{V}_\infty - \mathscr{V}} \mathbf{e}\,\mathrm{d}\upsilon. \tag{5.7}$$

The Green's integral $\mathcal{g}^{(n)}$ has the general forms

$$\mathcal{g}^{(n)} = \int_{\mathscr{S}} \boldsymbol{\psi}^{(n)}(\boldsymbol{\Omega} \wedge \boldsymbol{\Delta}\mathbf{v}) \cdot \mathbf{n}\, \mathrm{d}\mathcal{s}, \tag{5.8}$$

where the vorticity $\boldsymbol{\Omega}$ is spatially constant. Finally, from Appendix B, the zero- and first-order ambient force–moments are given by

$$\frac{1}{\rho^0}\mathbf{M}^{(0)} = \int_{\mathscr{V}} \frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}}\, \mathrm{d}\mathcal{v} \tag{5.9a}$$

and

$$\frac{1}{\rho^0}\mathbf{M}^{(1)} = \int_{\mathscr{V}} \left\{\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}} \otimes \tilde{\mathbf{x}} - \frac{1}{\rho^0}P\mathbf{I}\right\}\mathrm{d}\mathcal{v}. \tag{5.9b}$$

We can now derive the force $\mathbf{f}$ and torque $\mathbf{j}$ by noting that $\mathbf{f} = \mathbf{m}^{(0)}$ and $\mathbf{j} = \boldsymbol{\varepsilon} : \mathbf{m}^{(1)}$ and that double scalar pre-multiplication by the permutation tensor will cancel all symmetric tensors (namely **I** and **e**) and extract the anti-symmetric parts from the remaining tensors to give

$$\frac{1}{\rho^0}\mathbf{f} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{m}^{(0)} + \mathcal{m}^{(0)} \cdot (\mathbf{E} + \boldsymbol{\Xi}) + \int_{\mathscr{S}} \boldsymbol{\psi}^{(0)}(\boldsymbol{\Omega} \wedge \boldsymbol{\Delta}\mathbf{v}) \cdot \mathbf{n}\, \mathrm{d}\mathcal{s} + \frac{1}{\rho^0}\mathbf{F} \tag{5.10a}$$

and

$$\frac{1}{\rho^0}\mathbf{j} = \frac{\mathrm{d}}{\mathrm{d}t}\bar{\bar{\mathcal{m}}}^{(1)} + \mathfrak{J} + \int_{S} \bar{\bar{\boldsymbol{\psi}}}^{(1)}(\boldsymbol{\Omega} \wedge \boldsymbol{\Delta}\mathbf{v}) \cdot \mathbf{n}\, \mathrm{d}\mathcal{s} + \frac{1}{\rho^0}\mathbf{J}. \tag{5.10b}$$

Here the over double over bar represents the vector extraction of the anti-symmetric part of the corresponding tensor, namely $\bar{\bar{\mathbf{A}}}_{3\times1} = \boldsymbol{\varepsilon}_{3\times3\times3} : \mathbf{A}_{3\times3}$. $\mathfrak{J}$ is the lift-generated torque which is given in terms of the matrix commutator by

$$\mathfrak{J} = \mathcal{m}^{(0)} \wedge \mathbf{U}|_{\mathbf{0}} + \boldsymbol{\varepsilon} : [(\boldsymbol{\nabla} \otimes \mathbf{U})|_{\mathbf{0}}, \mathcal{m}^{(1)}] \tag{5.11a}$$

or equivalently in terms of the symmetric (superscript 's') and anti-symmetric (superscript 'a') parts of the strain-rate tensor and added mass tensor $\mathcal{m}^{(1)}$ as

$$\mathfrak{J} = \mathcal{m}^{(0)} \wedge \mathbf{U}|_{\mathbf{0}} + 2\boldsymbol{\varepsilon} : \mathbf{E} \cdot \mathcal{m}^{(1)\mathrm{s}} - 2\boldsymbol{\varepsilon} : \boldsymbol{\Xi} \cdot \mathcal{m}^{(1)\mathrm{a}} \tag{5.11b}$$

and further by noting that $\mathcal{m}^{(1)\mathrm{a}} = -\frac{1}{2}\boldsymbol{\varepsilon} \cdot \bar{\bar{\mathcal{m}}}^{(1)}$ then also

$$\mathfrak{J} = \mathcal{m}^{(0)} \wedge \mathbf{U}|_{\mathbf{0}} + 2\boldsymbol{\varepsilon} : \mathbf{E} \cdot \mathcal{m}^{(1)\mathrm{s}} + \frac{1}{2}\bar{\bar{\mathcal{m}}}^{(1)} \wedge \boldsymbol{\Omega}. \tag{5.11c}$$

F and **J** are the ambient force and torque as evaluated in Appendix B as

$$\frac{1}{\rho^0}\mathbf{F} = \int_{\mathscr{V}} \frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}}\, \mathrm{d}\mathcal{v}, \quad \frac{1}{\rho^0}\mathbf{J} = \int_{\mathscr{V}} \tilde{\mathbf{x}} \wedge \frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}}\, \mathrm{d}\mathcal{v}. \tag{5.12}$$

6. Comparison with Miloh and Galper & Miloh

Following Miloh (2003) we shall split the force and torque into their potential ($|_0$), linear vortical ($|_{\omega(1)}$) and quadratic vortical ($|_{\omega(2)}$) components. First split our disturbance velocity potential into its potential part $\Delta\varphi|_0$, Φ in Miloh's (2.11), and the vortical part $\Delta\varphi|_{\omega(1)}$ as

$$\Delta\varphi = \Delta\varphi|_0 + \Delta\varphi|_{\omega(1)} = \Delta\varphi|_0 + \boldsymbol{\psi}^{(1)} : \boldsymbol{\Xi}. \tag{6.1a}$$

The disturbance velocity then splits as

$$\boldsymbol{\Delta}\mathbf{v} = \boldsymbol{\Delta}\mathbf{v}|_0 + \boldsymbol{\Delta}\mathbf{v}|_{\omega(1)} = \boldsymbol{\nabla}\Delta\varphi|_0 + (\boldsymbol{\nabla}\otimes\boldsymbol{\psi}^{(1)}) : \boldsymbol{\Xi} \tag{6.1b}$$

and from our series expansion (2.14a) for the Kelvin Impulse then

$$\mathscr{m}^{(n)} = \mathscr{m}^{(n)}|_0 + \mathscr{m}^{(n)}|_{\omega(1)} = \int_{\mathscr{S}} \mathbf{n}\otimes\mathbf{x}^{(n)}\Delta\varphi|_0\,\mathrm{d}\mathscr{s} - \mathscr{m}^{(n,1)} : \boldsymbol{\Xi}. \tag{6.2}$$

Note from (6.1) and (6.2) that $\boldsymbol{\Delta}\mathbf{v}|_{\omega(1)} = (\boldsymbol{\nabla}\otimes\boldsymbol{\psi}^{(1)}) : \boldsymbol{\Xi}$, $\mathscr{m}^{(n)}|_{\omega(1)} = -\mathscr{m}^{(n,1)} : \boldsymbol{\Xi}$ and, therefore, $\bar{\bar{\mathscr{m}}}^{(1)}|_{\omega(1)} = -\boldsymbol{\varepsilon} : [\mathscr{m}^{(1,1)} : \boldsymbol{\Xi}]$. Our (5.10a) for the force then splits as

$$\frac{1}{\rho^0}\mathbf{f}|_0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{m}^{(0)}|_0 + \mathscr{m}^{(0)}|_0\cdot\mathbf{E} + \frac{1}{\rho^0}\mathbf{F}|_0, \tag{6.3a}$$

$$\frac{1}{\rho^0}\mathbf{f}|_{\omega(1)} = -\frac{\mathrm{d}}{\mathrm{d}t}[\mathscr{m}^{(0,1)} : \boldsymbol{\Xi}] + \mathscr{m}^{(0)}|_0\cdot\boldsymbol{\Xi} - \mathscr{m}^{(0,1)} : \boldsymbol{\Xi}\cdot\mathbf{E} + \int_{\mathscr{S}}\boldsymbol{\psi}^{(0)}(\boldsymbol{\Omega}\wedge\boldsymbol{\Delta}\mathbf{v}|_0)\cdot\mathbf{n}\,\mathrm{d}\mathscr{s} + \frac{1}{\rho^0}\mathbf{F}|_{\omega(1)}, \tag{6.3b}$$

$$\frac{1}{\rho^0}\mathbf{f}|_{\omega(2)} = -\mathscr{m}^{(0,1)} : \boldsymbol{\Xi}\cdot\boldsymbol{\Xi} + \int_{\mathscr{S}}\boldsymbol{\psi}^{(0)}[\boldsymbol{\Omega}\wedge(\boldsymbol{\nabla}\otimes\boldsymbol{\psi}^{(1)}) : \boldsymbol{\Xi}]\cdot\mathbf{n}\,\mathrm{d}\mathscr{s} \tag{6.3c}$$

and similarly our (5.10b) for the torque splits as

$$\frac{1}{\rho^0}\mathbf{j}|_0 = \frac{\mathrm{d}}{\mathrm{d}t}\bar{\bar{\mathscr{m}}}^{(1)}|_0 + \mathscr{m}^{(0)}|_0\wedge\mathbf{U}|_{\mathbf{0}} + 2\boldsymbol{\varepsilon} : \mathbf{E}\cdot\mathscr{m}^{(1)\mathrm{s}}|_0 + \frac{1}{\rho^0}\mathbf{J}|_0, \tag{6.4a}$$

$$\begin{aligned}\frac{1}{\rho^0}\mathbf{j}|_{\omega(1)} = {} & -\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\varepsilon} : \mathscr{m}^{(1,1)} : \boldsymbol{\Xi}] - [\mathscr{m}^{(0,1)} : \boldsymbol{\Xi}]\wedge\mathbf{U}|_{\mathbf{0}} + 2\boldsymbol{\varepsilon} : \mathbf{E}\cdot\mathscr{m}^{(1)\mathrm{s}}|_{\omega(1)} + \frac{1}{2}\bar{\bar{\mathscr{m}}}^{(1)}|_0\wedge\boldsymbol{\Omega} \\ & + \int_{\mathscr{S}}\bar{\bar{\boldsymbol{\psi}}}^{(1)}(\boldsymbol{\Omega}\wedge\boldsymbol{\Delta}\mathbf{v}|_0)\cdot\mathbf{n}\,\mathrm{d}\mathscr{s} + \frac{1}{\rho^0}\mathbf{J}|_{\omega(1)},\end{aligned} \tag{6.4b}$$

$$\frac{1}{\rho^0}\mathbf{j}|_{\omega(2)} = -\frac{1}{2}[\boldsymbol{\varepsilon} : \mathscr{m}^{(1,1)} : \boldsymbol{\Xi}]\wedge\boldsymbol{\Omega} + \int_{\mathscr{S}}\bar{\bar{\boldsymbol{\psi}}}^{(1)}[\boldsymbol{\Omega}\wedge(\boldsymbol{\nabla}\otimes\boldsymbol{\psi}^{(1)}) : \boldsymbol{\Xi}]\cdot\mathbf{n}\,\mathrm{d}\mathscr{s} + \frac{1}{\rho^0}\mathbf{J}|_{\omega(2)}. \tag{6.4c}$$

6.1. The vortical case

We now compare the above vortical force and torque components with Miloh (2003) by substituting from our (A.2), (A.3) and (A.7) for his added masses $\mathbf{Z}^{(\mathrm{M})}$, $\mathbf{R}^{(\mathrm{M})}$ and $\mathbf{j}^{(\mathrm{M})}$ (his symbols being denoted by the superscript M). Our variables are directly equated to Miloh's thus $\Delta\varphi|_0 \equiv \Phi^{(\mathrm{M})}$, $\boldsymbol{\psi}^{(0)} \equiv \boldsymbol{\varphi}^{(\mathrm{M})}$, $\bar{\bar{\boldsymbol{\psi}}}^{(1)} = \boldsymbol{\varepsilon} : \boldsymbol{\psi}^{(1)} \equiv \boldsymbol{\theta}^{(\mathrm{M})}$, $\boldsymbol{\Omega} \equiv \boldsymbol{\omega}_{\mathbf{o}}^{(\mathrm{M})}$. Noting that $\mathbf{n}\cdot\boldsymbol{\Xi} = -\frac{1}{2}\mathbf{n}\cdot\,\boldsymbol{\varepsilon}\cdot\boldsymbol{\Omega} = \frac{1}{2}\mathbf{n}\wedge\boldsymbol{\Omega}$ the linear vortical force becomes

$$\begin{aligned}\frac{1}{\rho^0}\mathbf{f}|_{\omega(1)} = {} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{Z}^{(\mathrm{M})}\cdot\boldsymbol{\Omega}] - \frac{1}{2}\boldsymbol{\Omega}\wedge\int_{\mathscr{S}}\Delta\varphi|_0\mathbf{n}\,\mathrm{d}\mathscr{s} + \frac{1}{2}[\mathbf{Z}^{(\mathrm{M})}\cdot\boldsymbol{\Omega}]\cdot\mathbf{E} \\ & + \int_{\mathscr{S}}(\boldsymbol{\Omega}\wedge\boldsymbol{\nabla}\Delta\varphi|_0\cdot\mathbf{n})\boldsymbol{\psi}^{(0)}\,\mathrm{d}\mathscr{s} + \frac{1}{\rho^0}\mathbf{F}|_{\omega(1)}.\end{aligned} \tag{6.5a}$$

Further since $(\nabla \otimes \boldsymbol{\psi}^{(1)}) : \boldsymbol{\Xi} = -\frac{1}{2}(\nabla \otimes \boldsymbol{\psi}^{(1)}) : (\boldsymbol{\varepsilon} \cdot \boldsymbol{\Omega}) = -\frac{1}{2}\nabla(\bar{\bar{\boldsymbol{\psi}}}^{(1)} \cdot \boldsymbol{\Omega})$ the quadratic vortical force becomes

$$\frac{1}{\rho^0}\mathbf{f}|_{\omega(2)} = -\frac{1}{4}\boldsymbol{\Omega} \wedge (\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}) - \frac{1}{2}\int_{\mathscr{S}} (\boldsymbol{\Omega} \wedge \nabla\left[\bar{\bar{\boldsymbol{\psi}}}^{(1)} \cdot \boldsymbol{\Omega}\right] \cdot \mathbf{n})\boldsymbol{\psi}^{(0)}\,\mathrm{d}\mathscr{s}. \tag{6.5b}$$

Finally, noting that $\bar{\bar{m}}^{(1)}|_0 = \int_{\mathscr{S}} \Delta\varphi|_0\, \boldsymbol{\varepsilon} : (\mathbf{n} \otimes \mathbf{x})\,\mathrm{d}\mathscr{s}$, the linear vortical torque becomes

$$\frac{1}{\rho^0}\mathbf{j}|_{\omega(1)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{R}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}] + \frac{1}{2}[\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}] \wedge \mathbf{U}|_{\mathbf{0}} + \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{E} \cdot [\boldsymbol{\Omega} \cdot \mathbf{j}^{(\mathrm{M})}] - \frac{1}{2}\boldsymbol{\Omega} \wedge \int_{\mathscr{S}} \Delta\varphi|_0 (\mathbf{x} \wedge \mathbf{n})\,\mathrm{d}\mathscr{s}$$
$$+ \int_{\mathscr{S}} (\boldsymbol{\Omega} \wedge \nabla\Delta\varphi|_0 \cdot \mathbf{n})\bar{\bar{\boldsymbol{\psi}}}^{(1)}\,\mathrm{d}\mathscr{s} + \frac{1}{\rho^0}\mathbf{J}|_{\omega(1)}, \tag{6.6a}$$

and the quadratic vortical torque

$$\frac{1}{\rho^0}\mathbf{j}|_{\omega(2)} = -\frac{1}{4}\boldsymbol{\Omega} \wedge [\mathbf{R}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}] - \frac{1}{2}\int_{\mathscr{S}} \left(\boldsymbol{\Omega} \wedge \nabla\left[\bar{\bar{\boldsymbol{\psi}}}^{(1)} \cdot \boldsymbol{\Omega}\right] \cdot \mathbf{n}\right)\bar{\bar{\boldsymbol{\psi}}}^{(1)}\,\mathrm{d}\mathscr{s} + \frac{1}{\rho^0}\mathbf{J}|_{\omega(2)}. \tag{6.6b}$$

Our comparison with Miloh's regular vortical terms $\mathbf{F}|_{\omega(1)}$, $\mathbf{J}|_{\omega(1)}$ and $\mathbf{J}|_{\omega(2)}$, is addressed in Appendix B. Note that since we express our ambient velocities relative to the inertial laboratory frame, then to compare with Miloh's results, the velocity of his body must be set to zero. The singular parts of our quadratic vortical force (6.5b) and torque (6.6b) agree identically with Miloh's (3.12) and (3.13). The singular parts of our linear vortical force (6.5a) and torque (6.6a) also agree with Miloh's (3.8) and (3.11) once the following differences between the two analyses are taken into account.

First, our transient force $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}]$ and torque $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{R}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}]$ will yield the first and second right-hand sides of Miloh's (3.8) and (3.11) if, like Miloh, we transform $\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}$ and $\mathbf{R}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}$ to the rotating frame as vectors, namely write $\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega} = \mathbf{Q} \cdot (\acute{\mathbf{Z}}^{(\mathrm{M})} \cdot \acute{\boldsymbol{\Omega}})$ and similarly for $\mathbf{R}^{(\mathrm{M})}$. Here the transformed ambient vorticity $\acute{\boldsymbol{\Omega}}$, where $\boldsymbol{\Omega} = \mathbf{Q} \cdot \acute{\boldsymbol{\Omega}}$, is a function of the body's rotation tensor $\mathbf{Q}$. This approach differs from our analysis in Section 2 where $\boldsymbol{\Omega}$ takes its value in the laboratory frame and the vector $\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}$ then transforms as a tensor thus $\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega} = \mathbf{Q}^{(2)} \cdot (\acute{\mathbf{Z}}^{(\mathrm{M})} \cdot \boldsymbol{\Omega})$.

Second, our singular steady linear vortical terms only differ from the fourth and fifth right-hand sides of Miloh's (3.8) and (3.11) in the contributions from infinity which he omitted to include and which result in our additional terms $\frac{1}{2}[\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}] \cdot \mathbf{E}$ in the force and $\frac{1}{2}[\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}] \wedge \mathbf{U}|_{\mathbf{0}} + \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{E} \cdot [\boldsymbol{\Omega} \cdot \mathbf{j}^{(\mathrm{M})}]$ in the torque. The contributions from infinity arise because the regular velocity $\mathbf{v}_{\mathrm{r}}^{(\mathrm{M})}$, in his (3.2), has the unbounded asymptotic form $\mathbf{v}_{\mathrm{r}}^{(\mathrm{M})} \sim \mathbf{E} \cdot \mathbf{x}$ as $|\mathbf{x}| \to +\infty$. Specifically, the missing terms arise in his identities (B.2) and (C.2) when applied to the functions $\nabla\phi^{(\mathrm{M})} = \nabla\varphi_{j+3}^{(\mathrm{M})}$ and $\nabla\psi^{(\mathrm{M})} = \nabla\Phi^{(\mathrm{M})} + \mathbf{v}_{\mathrm{r}}^{(\mathrm{M})}$.

6.2. The potential case

We shall compare our potential force $\mathbf{f}|_0$ and torque $\mathbf{j}|_0$, given by (6.3a) and (6.4a) above, with the analysis of G&M (Galper and Miloh, 1994) for general deforming bodies. Note that Miloh (2003, Appendix A) restricts his irrotational analysis to symmetric quadratic shapes. Since G&M

do not evaluate the deformation contribution to the force and torque explicitly the most direct comparison is to assume the body is rigid with rotation. We can then set G&M's deformation potential $\phi_{\mathrm{d}}^{(\mathrm{G\&M})}$ to zero. In our formulae the body's velocity $\mathbf{U}^{\mathrm{B}}$ consists only of the rotational component, thus $\mathbf{U}^{\mathrm{B}} = \boldsymbol{\Xi}^{\mathrm{B}} \cdot \mathbf{x}$, where the body's strain-rate $\boldsymbol{\Xi}^{\mathrm{B}}$ is related to the body's angular velocity $\dot{\boldsymbol{\Theta}}$ by $\boldsymbol{\Xi}^{\mathrm{B}} = -\boldsymbol{\varepsilon} \cdot \dot{\boldsymbol{\Theta}}$. Our difference velocity $\boldsymbol{\delta}\mathbf{U} = \mathbf{U} - \mathbf{U}^{\mathrm{B}}$ is then defined by the first two terms in its Taylor expansion thus

$$\boldsymbol{\delta}\mathbf{U}|_{\mathbf{0}} = \mathbf{U}|_{\mathbf{0}}; \quad (\boldsymbol{\nabla} \otimes \boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}} = (\mathbf{E} + \boldsymbol{\Xi}^{\mathrm{B}}) - \boldsymbol{\Xi}. \tag{6.7}$$

From our (2.14a), the potential part of the generalised Kelvin Impulse is

$$\mathscr{m}^{(n)}|_0 = \mathscr{m}^{(n,0)} \cdot \mathbf{U}|_{\mathbf{0}} + \mathscr{m}^{(n,1)} : \mathbf{E} + \mathscr{m}^{(n,1)} : \boldsymbol{\Xi}^{\mathrm{B}} \tag{6.8a}$$

which, by substituting from our (A.3)–(A.5), can be written in terms of G&M's tensors as

$$\mathscr{m}^{(0)}|_0 = \mathbf{T}^{(\mathrm{M})} \cdot \mathbf{U}|_{\mathbf{0}} - \frac{1}{2}\mathbf{s}^{(\mathrm{M})} : \mathbf{E} - \mathbf{Z}^{(\mathrm{M})} \cdot \dot{\boldsymbol{\Theta}}. \tag{6.8b}$$

Following G&M and dropping terms of the order of the strain-rate squared then

$$\mathscr{m}^{(0)}|_0 \cdot \mathbf{E} \approx (\mathbf{T}^{(\mathrm{M})} \cdot \mathbf{U}|_{\mathbf{0}} - \mathbf{Z}^{(\mathrm{M})} \cdot \dot{\boldsymbol{\Theta}}) \cdot \mathbf{E}. \tag{6.9}$$

Similarly, substituting from our (A.2), (A.3) and (A.6), the vector extraction of the anti-symmetric potential part of $\mathscr{m}^{(1)}$ is given by

$$\bar{\bar{\mathscr{m}}}^{(1)}|_0 = \boldsymbol{\varepsilon} : \mathscr{m}^{(1)}|_0 = \mathbf{U}|_{\mathbf{0}} \cdot \mathbf{Z}^{(\mathrm{M})} - \frac{1}{2}\mathbf{j}^{(\mathrm{M})} : \mathbf{E} - \mathbf{R}^{(\mathrm{M})} \cdot \dot{\boldsymbol{\Theta}}. \tag{6.10}$$

From out (A.11), dropping terms of the order of the strain-rate squared, then

$$\boldsymbol{\varepsilon} : \mathbf{E} \cdot \mathscr{m}^{(1)\mathrm{s}}|_0 \approx -\frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{E} \cdot (\mathbf{U}|_{\mathbf{0}} \cdot \mathbf{s}^{(\mathrm{M})} - \dot{\boldsymbol{\Theta}} \cdot \mathbf{j}^{(\mathrm{M})}). \tag{6.11}$$

Our (6.3a) and (6.4a) for the force and torque shall now be rearranged into G&M's steady and unsteady Lagally forces ($\mathbf{F}_{\mathrm{st}}^{(\mathrm{G\&M})}$, $\mathbf{F}_{\mathrm{un}}^{(\mathrm{G\&M})}$) and moments ($\mathbf{M}_{\mathrm{st}}^{(\mathrm{G\&M})}$, $\mathbf{M}_{\mathrm{un}}^{(\mathrm{G\&M})}$). Note that our absolute fluid velocity $\widetilde{\mathbf{U}}|_{\mathbf{0}}$, centroid velocity $\tilde{\mathbf{u}}^{\mathrm{B}}$ and angular velocity $\dot{\boldsymbol{\Theta}}$ are in G&M's notation given, respectively, by $\mathbf{V}_0^{(\mathrm{G\&M})}$, $\mathbf{U}^{(\mathrm{G\&M})}$ and $\boldsymbol{\Omega}^{(\mathrm{G\&M})}$. In addition our ambient force and torque ($\frac{1}{\rho^0}\mathbf{F}|_0$ and $\frac{1}{\rho^0}\mathbf{J}|_0$) are, in G&M's analysis, given by integrals involving their $\phi^{(\mathrm{G\&M})}$. Now split our $\frac{1}{\rho^0}\mathbf{F}|_0$ into G&M's unsteady ($\mathscr{V}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}}$) and steady ($\mathscr{V}\mathbf{E} \cdot \mathbf{U}|_{\tilde{\mathbf{x}}^{\mathrm{B}}}$) components (by, respectively, adding and subtracting $\mathscr{V}\mathbf{E} \cdot \tilde{\mathbf{u}}^{\mathrm{B}}$ to and from our unsteady and steady components), drop the term $(\boldsymbol{\varepsilon} \cdot [\mathbf{E} \cdot \mathbf{E}]) : \mathscr{V}^{(2)}$ in our ambient torque $\frac{1}{\rho^0}\mathbf{J}|_0$, which is of order the strain-rate squared, and combine the remaining term with the unsteady torque to give

$$\mathbf{F}_{\mathrm{un}}^{(\mathrm{G\&M})} = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{m}^{(0)}|_0 + \mathscr{V}\left(\frac{\partial}{\partial \tilde{t}}\widetilde{\mathbf{U}}\right)|_{\tilde{\mathbf{x}}^{\mathrm{B}}} + \mathscr{V}\mathbf{E} \cdot \tilde{\mathbf{u}}^{\mathrm{B}}, \quad \mathbf{F}_{\mathrm{st}}^{(\mathrm{G\&M})} = \mathscr{m}^{(0)}|_0 \cdot \mathbf{E} + \mathscr{V}\mathbf{E} \cdot \widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}} - \mathscr{V}\mathbf{E} \cdot \tilde{\mathbf{u}}^{\mathrm{B}} \tag{6.12a}$$

and

$$\mathbf{M}_{\mathrm{un}}^{(\mathrm{G\&M})} = \frac{\mathrm{d}}{\mathrm{d}t}\bar{\bar{\mathscr{m}}}^{(1)}|_0 + \left(\boldsymbol{\varepsilon} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\right) : \mathscr{V}^{(2)}, \quad \mathbf{M}_{\mathrm{st}}^{(\mathrm{G\&M})} = \mathscr{m}^{(0)}|_0 \wedge \mathbf{U}|_{\mathbf{0}} + 2\boldsymbol{\varepsilon} : \mathbf{E} \cdot \mathscr{m}^{(1)\mathrm{s}}|_0. \tag{6.12b}$$

Now substitute for the Kelvin Impulses from our (6.8)–(6.11) to give

$$\mathbf{F}_{\rm un}^{\rm (G\&M)} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{T}^{\rm (M)}\cdot\mathbf{U}|_{\mathbf{0}} - \frac{1}{2}\mathbf{s}^{\rm (M)} : \mathbf{E} - \mathbf{Z}^{\rm (M)}\cdot\dot{\mathbf{\Theta}}\right] + \mathscr{V}\frac{\mathrm{d}}{\mathrm{d}t}[\widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\rm B}}], \tag{6.13a}$$

$$\mathbf{F}_{\rm st}^{\rm (G\&M)} = (\mathbf{T}^{\rm (M)}\cdot\mathbf{U}|_{\mathbf{0}} - \mathbf{Z}^{\rm (M)}\cdot\dot{\mathbf{\Theta}})\cdot\mathbf{E} + \mathscr{V}\mathbf{E}\cdot\mathbf{U}|_{\mathbf{0}}, \tag{6.13b}$$

$$\mathbf{M}_{\rm un}^{\rm (G\&M)} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{U}|_{\mathbf{0}}\cdot\mathbf{Z}^{\rm (M)} - \frac{1}{2}\mathbf{j}^{\rm (M)} : \mathbf{E} - \mathbf{R}^{\rm (M)}\cdot\dot{\mathbf{\Theta}}\right] + \left(\boldsymbol{\varepsilon}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\right) : \mathscr{V}^{(2)}, \tag{6.14a}$$

$$\mathbf{M}_{\rm st}^{\rm (G\&M)} = \left[\mathbf{T}^{\rm (M)}.\mathbf{U}|_{\mathbf{0}} - \frac{1}{2}\mathbf{s}^{\rm (M)} : \mathbf{E} - \mathbf{Z}^{\rm (M)}\cdot\dot{\mathbf{\Theta}}\right]\wedge\mathbf{U}|_{\mathbf{0}} - \boldsymbol{\varepsilon} : \mathbf{E}\cdot(\mathbf{U}|_{\mathbf{0}}\cdot\mathbf{s}^{\rm (M)} - \dot{\mathbf{\Theta}}\cdot\mathbf{j}^{\rm (M)}). \tag{6.14b}$$

To avoid confusion when comparing G&M's differentiated-out forms for the unsteady terms, which depend upon the tensor reference frame, we consider only the earlier part of G&M's analysis.

The first right-hand sides (r.h.s.') of our (6.13a) and (6.14a) for the unsteady Lagally force and torque are equal, respectively, to the sum of the first and second r.h.s.' of G&M's (3.13) for the force and the sum of the l.h.s.' of G&M's (4.9), (4.10) and (4.19) for the torque. Note that the second r.h.s. of our (6.13a) for the unsteady ambient force agrees with the third r.h.s. of G&M's (3.13) which is also given by combining their (3.14) and (3.19). Note that the second r.h.s of G&M's (3.19) does not appear here since we are assuming the body is rigid. The second r.h.s. of our (6.14a) for the unsteady ambient torque equals the time derivative of their (4.11) when the body's volume is constant.

Now compare the steady Lagally force and torque components. Our steady force (6.13b) is seen to agree with G&M's (3.12) by noting that their first and third r.h.s.' are zero when the body is rigid. Similarly, our steady torque (6.14b) can be compared with G&M's (4.8) once the rigid body assumption is imposed by setting their surface deformation $\dot{S}$ to zero. Note that since their tensors $\mathbf{J}^{\rm (G\&M)}$ and $\mathbf{e}^{\rm (G\&M)}$ are symmetric and $\mathbf{\Omega}^{\rm (G\&M)}$ is anti-symmetric their ambient contribution $\tau^{-1}[[\mathbf{\Omega}^{\rm (G\&M)}, \mathbf{J}^{\rm (G\&M)}], \mathbf{e}^{\rm (G\&M)}]$ is zero. Also note that the second line of their (4.8) (and their A.16) is misprinted since these terms do not arise from the sum of their (A.7), (A.14) and (A.15). Agreement with G&M's steady torque now follows by noting that the second r.h.s. of our (6.14b) is identically equal to their $-\tau^{-1}([(\mathbf{V}_0^{\rm (G\&M)} - \mathbf{U}^{\rm (G\&M)})\cdot\mathbf{s}^{\rm (M)} + \mathbf{\Omega}^{\rm (G\&M)}\cdot\mathbf{j}^{\rm (M)}, \mathbf{e}^{\rm (G\&M)}])$.

7. Discussion

This paper has presented the generalised force–moment equations for an arbitrarily shaped deforming body, given by the sum of our (3.1), (3.6) and (4.6) as

$$\frac{1}{\rho^0}\mathbf{m}^{(n)} = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{m}^{(n)} + \mathscr{L}^{(n)} - \mathsf{D}^{(n)} - \mathscr{E}^{(n)} + \mathscr{g}^{(n)} + \frac{1}{\rho^0}\mathbf{M}^{(n)}. \tag{7.1}$$

This infinite set of equations apply to a general deforming body introduced impulsively into an ideal incompressible fluid whose velocity field has arbitrary spatial gradients provided that the vorticity is spatially uniform. In the absence of vorticity the equations apply at all times and, therefore, describe the dynamics of bubbles in arbitrary irrotational flow fields.

The above equation set, and its generalisations, also provide a theoretical foundation for new analytical and numerical approaches. In particular, the grad operator in our (4.8) for the generalised lift can, in principle, be replaced by its surface tangent form $\boldsymbol{\nabla}^{\tau}(=\boldsymbol{\nabla}-\mathbf{n}\mathbf{n}\cdot\boldsymbol{\nabla})$ thus allowing the arguments in Section 2–4 to be restricted to the body's surface. $\mathbf{U}^{\mathrm{B}}$ then would no longer need an analytic continuation into the interior of the body. Importantly, the tensor power $\mathbf{x}^{(n)}$ could be replaced throughout by the nth scalar basis function, $\chi^{(n)}$ say, of any complete orthonormal set of functions defined on the body's surface. Our dynamical equations would then become an infinite set of vector equations and our Kirchhoff potentials further generalised to vector functions satisfying $\frac{\partial}{\partial n}\boldsymbol{\psi}^{(n)}|_{\mathscr{S}}=\chi^{(n)}\mathbf{n}|_{\mathscr{S}}$.

Whilst applications of the equations lie outside the scope of this paper it is nonetheless instructive to explore the mathematical nature of the general problem. We shall confine our discussion to the case of a gas bubble in a relatively much denser liquid when inertial forces due to the dynamics of the bubble gas can be neglected. In this case the force–moment $\mathbf{m}^{(n)}$ is a function only of the shape of the bubble and the surface tension. We can simplify the problem further if we apply our equations to the dilute dispersed gas phase of a bubbly liquid by assuming the liquid phase dynamics are fully prescribed and, consequently, our ambient force moments $\mathbf{M}^{(n)}$ are prescribed. The dispersed phase bubbles then satisfy equations of the following form where the right-hand side now takes the form of a forcing function.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{m}^{(n)}+\mathscr{L}^{(n)}-\mathsf{D}^{(n)}-\mathscr{E}^{(n)}=\frac{1}{\rho^{0}}\mathbf{m}^{(n)}-\frac{1}{\rho^{0}}\mathbf{M}^{(n)}. \tag{7.2}$$

By substituting our expansion (2.14a) for the Kelvin Impulse $\mathscr{m}^{(n)}$ we obtain a system of simultaneous linear equations for the generalised difference accelerations $\frac{\mathrm{d}}{\mathrm{d}t}[(\boldsymbol{\nabla}^{(m)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}]$. Note that the difference velocity $\boldsymbol{\delta}\mathbf{U}(=\mathbf{U}-\mathbf{U}^{\mathrm{B}})$ is defined as the difference between the prescribed fluid velocity $\mathbf{U}$ and the unknown continuum velocity $\mathbf{U}^{\mathrm{B}}$ of the body. Further note that the generalised lift $\mathscr{L}^{(n)}$, as shown in Appendix C, is a linear function of the added masses and, therefore, is also linear in the generalised difference velocities $(\boldsymbol{\nabla}^{(m)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}$. By virtue of the expansions given in Appendix D the surface momentum flux $\mathsf{D}^{(n)}$ and surface kinetic energy $\mathscr{E}^{(n)}$ are both quadratic in the deformation velocities $(\boldsymbol{\nabla}^{(m)}\otimes\mathbf{U}^{\mathrm{B}})|_{\mathbf{0}}$, since by inspection of (D.3) and (D.5) the double series expansion for $\mathsf{D}^{(n)}$ involves the terms $(\boldsymbol{\nabla}^{(m)}\otimes\mathbf{U}^{\mathrm{B}})|_{\mathbf{0}}\otimes(\boldsymbol{\nabla}^{(p)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}$ and $\mathscr{E}^{(n)}$ the terms $(\boldsymbol{\nabla}^{(m)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}\otimes(\boldsymbol{\nabla}^{(p)}\otimes\boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}$. Importantly the geometrical dependencies in the inertial $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{m}^{(n)}$, lift $\mathscr{L}^{(n)}$ and surface momentum flux $\mathsf{D}^{(n)}$ have the form of linear combinations of coefficients of the generalised added mass tensor $\mathscr{m}^{(n,m)}$ whereas the surface kinetic energy $\mathscr{E}^{(n)}$ involves independent geometric quantities, the added Kirchhoff energies $\mathscr{E}^{(n,m,p)}$. Finally, in the quasi-steady regime of bubble deformation in which the inertial terms $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{m}^{(n)}$ can be neglected (physically defined as when the time-scales of changes in the ambient fluid are long compared to the natural response time of the bubbles) then the bubble's generalised deformation velocities $(\boldsymbol{\nabla}^{(m)}\otimes\mathbf{U}^{\mathrm{B}})|_{\mathbf{0}}$ are determined by an algebraic system of simultaneous quadratic equations.

Appendix A. Relationships between our $\mathscr{m}^{(n,m)}$ and Miloh's added mass tensors

Miloh (2003, Appendix A) defines the following three second-order added mass tensors

$$\mathbf{T}^{(\mathrm{M})}=\int_{\mathscr{S}}-\boldsymbol{\varphi}^{(\mathrm{M})}\otimes\frac{\partial}{\partial n}\boldsymbol{\varphi}^{(\mathrm{M})}\,\mathrm{d}s,\quad \mathbf{Z}^{(\mathrm{M})}=\int_{\mathscr{S}}-\boldsymbol{\varphi}^{(\mathrm{M})}\otimes\boldsymbol{\theta}^{(\mathrm{M})}\,\mathrm{d}s,\quad \mathbf{R}^{(\mathrm{M})}=\int_{\mathscr{S}}-\boldsymbol{\theta}^{(\mathrm{M})}\otimes\boldsymbol{\theta}^{(\mathrm{M})}\,\mathrm{d}s. \tag{A.1}$$

His Kirchhoff potential vectors $\boldsymbol{\varphi}^{(\mathrm{M})}(M) = (\varphi_1, \varphi_2, \varphi_3)$ and $\boldsymbol{\theta}^{(\mathrm{M})} = (\varphi_4, \varphi_5, \varphi_6)$ satisfy the boundary conditions $\frac{\partial}{\partial n}\boldsymbol{\varphi}^{(\mathrm{M})} = \mathbf{n}|_{\mathscr{S}}$ and $\frac{\partial}{\partial n}\boldsymbol{\theta}^{(\mathrm{M})} = \mathbf{x} \wedge \mathbf{n}|_{\mathscr{S}} = \boldsymbol{\varepsilon} : (\mathbf{n} \otimes \mathbf{x})|_{\mathscr{S}}$ and, therefore, are related to our $\boldsymbol{\psi}^{(0)}$ and $\boldsymbol{\psi}^{(1)}$ through $\boldsymbol{\varphi}^{(\mathrm{M})} = \boldsymbol{\psi}^{(0)}$ and $\boldsymbol{\theta}^{(\mathrm{M})} = \bar{\bar{\boldsymbol{\psi}}}^{(1)} = \boldsymbol{\varepsilon} : \boldsymbol{\psi}^{(1)}$. Miloh's added masses are then related to our generalised added masses $\mathscr{m}^{(m,n)}$, as given by our (2.14b), through

$$\mathbf{T}^{(\mathrm{M})} = \int_{\mathscr{S}} -\boldsymbol{\psi}^{(0)} \otimes \frac{\partial}{\partial n}\boldsymbol{\psi}^{(0)}\,\mathrm{d}s = \mathscr{m}^{(0,0)}, \tag{A.2a}$$

$$\mathbf{Z}^{(\mathrm{M})} = \int_{\mathscr{S}} -\boldsymbol{\psi}^{(0)} \otimes \left(\boldsymbol{\varepsilon} : \frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)}\right)\mathrm{d}s = \mathscr{m}^{(0,1)} : \boldsymbol{\varepsilon}, \tag{A.2b}$$

$$[\mathbf{Z}^{(\mathrm{M})}]^{\mathrm{T}} = \int_{\mathscr{S}} -(\boldsymbol{\varepsilon} : \boldsymbol{\psi}^{(1)}) \otimes \frac{\partial}{\partial n}\boldsymbol{\psi}^{(0)}\,\mathrm{d}s = \boldsymbol{\varepsilon} : \mathscr{m}^{(1,0)}, \tag{A.2c}$$

$$\mathbf{R}^{(\mathrm{M})} = \int_{\mathscr{S}} -(\boldsymbol{\varepsilon} : \boldsymbol{\psi}^{(1)}) \otimes \left(\boldsymbol{\varepsilon} : \frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)}\right)\mathrm{d}s = \boldsymbol{\varepsilon} : \mathscr{m}^{(1,1)} : \boldsymbol{\varepsilon}. \tag{A.2d}$$

Relationships that follow immediately from the above are:

$$\mathscr{m}^{(0,1)} : \boldsymbol{\Xi} = -\frac{1}{2}\mathscr{m}^{(0,1)} : (\boldsymbol{\varepsilon} \cdot \boldsymbol{\Omega}) = -\frac{1}{2}\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}, \tag{A.3a}$$

$$\boldsymbol{\varepsilon} : \mathscr{m}^{(1,1)} : \boldsymbol{\Xi} = -\frac{1}{2}\boldsymbol{\varepsilon} : \mathscr{m}^{(1,1)} : (\boldsymbol{\varepsilon} \cdot \boldsymbol{\Omega}) = -\frac{1}{2}\mathbf{R}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}, \tag{A.3b}$$

$$\mathscr{m}^{(0,1)} : \boldsymbol{\Xi} \cdot \boldsymbol{\Xi} = \frac{1}{4}(\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}).(\boldsymbol{\varepsilon} \cdot \boldsymbol{\Omega}) = \frac{1}{4}\boldsymbol{\Omega} \wedge (\mathbf{Z}^{(\mathrm{M})} \cdot \boldsymbol{\Omega}). \tag{A.3c}$$

Miloh's (2003) and Galper and Miloh's (1994) results also involve the following third-order tensors, defined in terms of our generalised Kirchhoff potentials by:

$$\mathbf{s}^{(\mathrm{M})} = \int_{\mathscr{S}}^{(0)} \boldsymbol{\psi}^{(0)} \otimes \left(\frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)} + \frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)^{\mathrm{T}}}\right)\mathrm{d}s \tag{A.4a}$$

and

$$\mathbf{j}^{(\mathrm{M})} = \int_{\mathscr{S}} (\boldsymbol{\varepsilon} : \boldsymbol{\psi}^{(1)}) \otimes \left(\frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)} + \frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)^{\mathrm{T}}}\right)\mathrm{d}s. \tag{A.4b}$$

Due to the symmetry of the strain-rate tensor $\mathbf{E}$ we have $\frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)} : \mathbf{E} = \frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)^{\mathrm{T}}} : \mathbf{E} = \frac{1}{2}\left(\frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)} + \frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)^{\mathrm{T}}}\right) : \mathbf{E}$. It follows, in agreement with (3.15) and (4.19) of Galper and Miloh (1994), that

$$\mathscr{m}^{(0,1)} : \mathbf{E} = \int_{\mathscr{S}} -\boldsymbol{\psi}^{(0)} \otimes \frac{\partial}{\partial n}\boldsymbol{\psi}^{(1)} : \mathbf{E}\,\mathrm{d}s = -\frac{1}{2}\mathbf{s}^{(\mathrm{M})} : \mathbf{E} \tag{A.5}$$

and

$$\boldsymbol{\varepsilon} : \mathscr{m}^{(1,1)} : \mathbf{E} = \int_{\mathscr{S}} -\boldsymbol{\varepsilon} : \boldsymbol{\psi}^{(1)} \otimes \frac{\partial}{\partial n} \boldsymbol{\psi}^{(1)} : \mathbf{E} \, \mathrm{d}s = -\frac{1}{2} \mathbf{j}^{(\mathrm{M})} : \mathbf{E}. \tag{A.6}$$

For the linear vortical part of $\mathscr{m}^{(1)\mathrm{s}}$ we have

$$\mathscr{m}^{(1)\mathrm{s}}|_{\omega(1)} = -\frac{1}{2}\left([\mathscr{m}^{(1,1)} : \boldsymbol{\Xi}] + [\mathscr{m}^{(1,1)} : \boldsymbol{\Xi}]^{\mathrm{T}}\right) = -\frac{1}{4} \int_{\mathscr{S}} (\boldsymbol{\psi}^{(1)} + \boldsymbol{\psi}^{(1)^{\mathrm{T}}}) \otimes \frac{\partial}{\partial n} \boldsymbol{\psi}^{(1)} : (\boldsymbol{\varepsilon} \cdot \boldsymbol{\Omega}) \, \mathrm{d}s. \tag{A.7a}$$

Reordering the integrand and applying Green's theorem

$$\mathscr{m}^{(1)\mathrm{s}}|_{\omega(1)} = \frac{1}{4} \int_{\mathscr{S}} \left(\boldsymbol{\Omega} \cdot \boldsymbol{\varepsilon} : \frac{\partial}{\partial n}^{(1)} \right) \otimes (\boldsymbol{\psi}^{(1)} + \boldsymbol{\psi}^{(1)^{\mathrm{T}}}) \, \mathrm{d}s = \frac{1}{4} \boldsymbol{\Omega} \cdot \mathbf{j}^{(\mathrm{M})}. \tag{A.7b}$$

It remains to consider the potential part of $\mathscr{m}^{(1)\mathrm{s}}$ for solid bodies in an irrotational ambient flow with uniform strain-rate. In this case the difference velocity $\boldsymbol{\delta}\mathbf{U}$ satisfies $\boldsymbol{\delta}\mathbf{U}|_{\mathbf{0}} = \mathbf{U}|_{\mathbf{0}}$ and $\nabla \otimes \boldsymbol{\delta}\mathbf{U}|_{\mathbf{0}} = \mathbf{E} + \boldsymbol{\Xi}^{\mathrm{B}}$ so from (2.14a)

$$\mathscr{m}^{(1)\mathrm{s}}|_{0} = \frac{1}{2}\left([\mathscr{m}^{(1,0)} \cdot \mathbf{U}|_{\mathbf{0}}] + [\mathscr{m}^{(1,0)} \cdot \mathbf{U}|_{\mathbf{0}}]^{\mathrm{T}}\right) + \frac{1}{2}\left([\mathscr{m}^{(1,1)} : (\mathbf{E} + \boldsymbol{\Xi}^{\mathrm{B}})] + [\mathscr{m}^{(1,1)} : (\mathbf{E} + \boldsymbol{\Xi}^{\mathrm{B}})]^{\mathrm{T}}\right). \tag{A.8}$$

From Green's theorem

$$\mathscr{m}^{(1,0)} \cdot \mathbf{U}|_{\mathbf{0}} = \int_{\mathscr{S}} -\boldsymbol{\psi}^{(1)} \otimes \frac{\partial}{\partial n} \boldsymbol{\psi}^{(0)} . \mathbf{U}|_{\mathbf{0}} \, \mathrm{d}s = \mathbf{U}|_{\mathbf{0}} \cdot \int_{\mathscr{S}} -\boldsymbol{\psi}^{(0)} \otimes \frac{\partial}{\partial n} \boldsymbol{\psi}^{(1)} \, \mathrm{d}s \tag{A.9a}$$

and thus

$$\frac{1}{2}\left([\mathscr{m}^{(1,0)} \cdot \mathbf{U}|_{\mathbf{0}}] + [\mathscr{m}^{(1,0)} \cdot \mathbf{U}|_{\mathbf{0}}]^{\mathrm{T}}\right) = \frac{1}{2} \mathbf{U}|_{\mathbf{0}} \cdot \int_{\mathscr{S}} -\boldsymbol{\psi}^{(0)} \otimes \frac{\partial}{\partial n} (\boldsymbol{\psi}^{(1)} + \boldsymbol{\psi}^{(1)^{\mathrm{T}}}) \, \mathrm{d}s = -\frac{1}{2} \mathbf{U}|_{\mathbf{0}} \cdot \mathbf{s}^{(\mathrm{M})}. \tag{A.9b}$$

Similarly

$$\frac{1}{2}\left([\mathscr{m}^{(1,1)} : (\mathbf{E} + \boldsymbol{\Xi}^{\mathrm{B}})] + [\mathscr{m}^{(1,1)} : (\mathbf{E} + \boldsymbol{\Xi}^{\mathrm{B}})]^{\mathrm{T}}\right) = -\frac{1}{2} (\mathbf{E} + \boldsymbol{\Xi}^{\mathrm{B}}) : \mathbf{k} \tag{A.10a}$$

where $\mathbf{k}$ is our fourth-order tensor defined by

$$\mathbf{k} = \int_{\mathscr{S}} \boldsymbol{\psi}^{(1)} \otimes \frac{\partial}{\partial n} (\boldsymbol{\psi}^{(1)} + \boldsymbol{\psi}^{(1)^{\mathrm{T}}}) \, \mathrm{d}s. \tag{A.10b}$$

Noting that $\boldsymbol{\Xi}^{\mathrm{B}} : \mathbf{k} = -(\boldsymbol{\varepsilon}.\dot{\boldsymbol{\Theta}}) : \mathbf{k} = -\dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\varepsilon} : \mathbf{k}) = -\dot{\boldsymbol{\Theta}} \cdot \mathbf{j}^{(\mathrm{M})}$ it follows, by substituting (A.9) and (A.10) into (A.8), that

$$\mathscr{m}^{(1)\mathrm{s}}|_{0} = -\frac{1}{2} \mathbf{U}|_{\mathbf{0}} \cdot \mathbf{s}^{(\mathrm{M})} - \frac{1}{2} \mathbf{E} : \mathbf{k} + \frac{1}{2} \dot{\boldsymbol{\Theta}} \cdot \mathbf{j}^{(\mathrm{M})}. \tag{A.11}$$

Appendix B. The ambient force–moments and comparison with Miloh

The ambient force–moments $\mathbf{M}^{(n)}$ are defined in terms of the ambient pressure P by

$$\frac{1}{\rho^0}\mathbf{M}^{(n)} = \int_{\mathscr{S}} -\frac{1}{\rho^0} P\,\mathbf{n}\otimes\mathbf{x}^{(n)}\,\mathrm{d}s. \tag{B.1}$$

Without loss of generality we take the origins of our inertial (laboratory) and non-inertial (centroid fixed) coordinate systems to be instantaneously coincident. Thus $\mathbf{x}^{(n)} = \tilde{\mathbf{x}}^{(n)}$ and, therefore, $\mathbf{\nabla}^{(n)}\otimes\mathbf{U} = \widetilde{\mathbf{\nabla}}^{(n)}\otimes\widetilde{\mathbf{U}}$. The divergence theorem can now be applied to the body's volume $\mathscr{V}$, because of the regularity of P within $\mathscr{V}$, to give

$$\frac{1}{\rho^0}\mathbf{M}^{(n)} = \int_{\mathscr{V}} -\frac{1}{\rho^0}\widetilde{\mathbf{\nabla}}\otimes(P\tilde{\mathbf{x}}^{(n)})\,\mathrm{d}v = \int_{\mathscr{V}} -\frac{1}{\rho^0}\{\widetilde{\mathbf{\nabla}}P\otimes\tilde{\mathbf{x}}^{(n)} + P\widetilde{\mathbf{\nabla}}\otimes\tilde{\mathbf{x}}^{(n)}\}\,\mathrm{d}v \tag{B.2}$$

and $\widetilde{\mathbf{\nabla}}P$ eliminated by substituting the momentum equation $\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}} + \frac{1}{\rho^0}\widetilde{\mathbf{\nabla}}P = 0$ thus

$$\frac{1}{\rho^0}\mathbf{M}^{(n)} = \int_{\mathscr{V}}\left\{\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}}\otimes\tilde{\mathbf{x}}^{(n)} - \frac{1}{\rho^0}P\widetilde{\mathbf{\nabla}}\otimes\tilde{\mathbf{x}}^{(n)}\right\}\mathrm{d}v. \tag{B.3}$$

In particular, the first- and second-order ambient force–moments are given by

$$\frac{1}{\rho^0}\mathbf{F} = \frac{1}{\rho^0}\mathbf{M}^{(0)} = \int_{\mathscr{V}}\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}}\,\mathrm{d}v, \tag{B.4}$$

$$\frac{1}{\rho^0}\mathbf{M}^{(1)} = \int_{\mathscr{V}}\left\{\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}}\otimes\tilde{\mathbf{x}} - \frac{1}{\rho^0}P\mathbf{I}\right\}\mathrm{d}v \tag{B.5}$$

and the torque $\mathbf{J}$ is obtained by pre-multiplying $\mathbf{M}^{(1)}$ by $\boldsymbol{\varepsilon}$: to give

$$\frac{1}{\rho^0}\mathbf{J} = \int_{\mathscr{V}}\tilde{\mathbf{x}}\wedge\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}}\,\mathrm{d}v = \int_{\mathscr{V}}\boldsymbol{\varepsilon}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}}\cdot\tilde{\mathbf{x}}\,\mathrm{d}v. \tag{B.6}$$

To compare with Miloh (2003) we assume the fluid velocity to have spatially uniform strain-rate, thus

$$\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}|_{\tilde{\mathbf{0}}} + (\widetilde{\mathbf{\nabla}}\otimes\widetilde{\mathbf{U}})^{\mathrm{T}}\cdot\tilde{\mathbf{x}} \quad \text{and} \quad \frac{\partial}{\partial\tilde{t}}(\widetilde{\mathbf{\nabla}}\otimes\widetilde{\mathbf{U}})^{\mathrm{T}} = \frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\mathbf{\nabla}}\otimes\widetilde{\mathbf{U}})^{\mathrm{T}}. \tag{B.7}$$

The Taylor expansion of the Lagrangian acceleration $\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}}$ is then

$$\frac{\mathrm{D}}{\mathrm{D}t}\widetilde{\mathbf{U}} = \left(\frac{\partial}{\partial\tilde{t}}\widetilde{\mathbf{U}}\right)\Big|_{\tilde{\mathbf{x}}^{\mathrm{B}}} + \frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\mathbf{\nabla}}\otimes\widetilde{\mathbf{U}})^{\mathrm{T}}\cdot\tilde{\mathbf{x}} + (\widetilde{\mathbf{\nabla}}\otimes\widetilde{\mathbf{U}})^{\mathrm{T}}\cdot\widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}} + (\widetilde{\mathbf{\nabla}}\otimes\widetilde{\mathbf{U}})^{\mathrm{T}}\cdot(\widetilde{\mathbf{\nabla}}\otimes\widetilde{\mathbf{U}})^{\mathrm{T}}\cdot\tilde{\mathbf{x}}. \tag{B.8}$$

Substituting (B.8) into (B.4) and (B.6) and dropping terms proportional to $\tilde{\mathbf{x}}$ because the body centroid is at the origin, we then find

$$\frac{1}{\rho^0}\mathbf{F} = \mathscr{V}\left[\left(\frac{\partial}{\partial\tilde{t}}\widetilde{\mathbf{U}}\right)\Big|_{\tilde{\mathbf{x}}^{\mathrm{B}}} + (\widetilde{\mathbf{\nabla}}\otimes\widetilde{\mathbf{U}})^{\mathrm{T}}\cdot\widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}}\right] \tag{B.9}$$

and when expressed in terms of the second moment of volume $\mathscr{V}^{(2)}$

$$\frac{1}{\rho^0}\mathbf{J} = \left\{ \boldsymbol{\varepsilon} \cdot \left[\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\mathbf{\nabla}} \otimes \widetilde{\mathbf{U}})^{\mathrm{T}} + (\widetilde{\mathbf{\nabla}} \otimes \widetilde{\mathbf{U}})^{\mathrm{T}} \cdot (\widetilde{\mathbf{\nabla}} \otimes \widetilde{\mathbf{U}})^{\mathrm{T}} \right] \right\} : \mathscr{V}^{(2)}. \tag{B.10}$$

Writing the strain-rate in terms of its symmetric and anti-symmetric parts, whilst noting that $\mathbf{\Xi}.\widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}} = -\frac{1}{2}\widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}} \wedge \mathbf{\Omega}$, the ambient force and torque split into the following potential and vortical components. For the force

$$\frac{1}{\rho^0}\mathbf{F}|_0 = \mathscr{V}\left[\left(\frac{\partial}{\partial \tilde{t}}\widetilde{\mathbf{U}}\right)|_{\tilde{\mathbf{x}}^{\mathrm{B}}} + \mathbf{E} \cdot \widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}}\right], \quad \frac{1}{\rho^0}\mathbf{F}|_{\omega(1)} = -\frac{1}{2}\mathscr{V}\widetilde{\mathbf{U}}|_{\tilde{\mathbf{x}}^{\mathrm{B}}} \wedge \mathbf{\Omega} \tag{B.11}$$

and for the torque

$$\frac{1}{\rho^0}\mathbf{J}|_0 = \left\{ \boldsymbol{\varepsilon} \cdot \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E} + \mathbf{E} \cdot \mathbf{E} \right] \right\} : \mathscr{V}^{(2)}, \tag{B.12a}$$

$$\frac{1}{\rho^0}\mathbf{J}|_{\omega(1)} = \left\{ \boldsymbol{\varepsilon} \cdot \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Xi} + [\mathbf{E} \cdot \mathbf{\Xi} + \mathbf{\Xi} \cdot \mathbf{E}] \right] \right\} : \mathscr{V}^{(2)}, \tag{B.12b}$$

$$\frac{1}{\rho^0}\mathbf{J}|_{\omega(2)} = \{ \boldsymbol{\varepsilon} \cdot [\mathbf{\Xi} \cdot \mathbf{\Xi}] \} : \mathscr{V}^{(2)} = -\frac{1}{4}\mathbf{\Omega} \wedge [\mathscr{V}^{(2)} \cdot \mathbf{\Omega}]. \tag{B.12c}$$

Comparison of our linear vortical torque $\mathbf{J}|_{\omega(1)}$ with Miloh's $\mathbf{K}^{(\mathrm{M})}$ is made more direct by substituting $\mathbf{\Xi} = -\frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbf{\Omega}$ and writing in terms of vector products as

$$\mathbf{J}|_{\omega(1)} = \frac{1}{2}\int_{\mathscr{V}} \tilde{\mathbf{x}} \wedge \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Omega} \wedge \tilde{\mathbf{x}} + \mathbf{\Omega} \wedge (\mathbf{E} \cdot \tilde{\mathbf{x}}) - \mathbf{E} \cdot (\tilde{\mathbf{x}} \wedge \mathbf{\Omega}) \right] \mathrm{d}\upsilon. \tag{B.13}$$

Our regular vortical force (B.11) and regular linear vortical torque (B.13) then agree, respectively, with the right-hand side of Miloh's (3.6) and his $\mathbf{K}^{(\mathrm{M})}$ after his (3.11). Note that his far right-hand term $[\boldsymbol{\omega}_{\mathbf{o}}^{(\mathrm{M})} \wedge \mathbf{r}] \wedge (\mathbf{E} \cdot \mathbf{r})$ in $\mathbf{K}^{(\mathrm{M})}$ is misprinted and should read $[\boldsymbol{\omega}_{\mathbf{o}}^{(\mathrm{M})} \wedge (\mathbf{E} \cdot \mathbf{r})] \wedge \mathbf{r}$. Our regular quadratic vortical torque (B.12c) agrees with the second right-hand term in Miloh's (3.13).

Appendix C. The generalised lift $\mathscr{L}^{(n)}$ expressed in terms of Kelvin Impulses

In index notation our (4.8) for our generalised lift $\mathscr{L}^{(n)}$ is

$$\mathscr{L}_k^{(n)} = \int_{\mathscr{S}} (\langle \mathbf{n}, \mathbf{\Delta v} \rangle_{k\ell} U_\ell) \otimes \mathbf{x}^{(n)} \,\mathrm{d}\jmath. \tag{C.1}$$

The tensor commutator in the integrand has the alternative forms

$$\langle \mathbf{n}, \mathbf{\Delta v} \rangle_{k\ell} = \langle n_k \Delta v_\ell - n_\ell \Delta v_k \rangle = n_p \nabla_q \langle\langle qp|k\ell \rangle\rangle = (\mathbf{n} \otimes \mathbf{\nabla}\Delta\varphi) : \mathbf{\Pi} \tag{C.2a}$$

in terms of the two following fourth-order tensors

$$\langle\langle qp|k\ell \rangle\rangle = \langle\langle \Delta\varphi\, \mathbf{\Pi}_{qpk\ell} \rangle\rangle, \quad \mathbf{\Pi}_{qpk\ell} = (\delta_{q\ell}\delta_{pk} - \delta_{qk}\delta_{p\ell}). \tag{C.2b}$$

For any tensor function, $\mathscr{F}$, therefore, the following identities hold

$$\nabla_p \nabla_q [\mathscr{F} \langle\langle qp|k\ell \rangle\rangle] = \mathbf{0} \quad \text{and} \quad [\nabla_p \nabla_q \mathscr{F}] \langle\langle qp|k\ell \rangle\rangle = \mathbf{0} \tag{C.3}$$

since $\langle\langle qp|k\ell\rangle\rangle$ is anti-symmetric in the indices (p, q). The integrand of (C.1) can now be written

$$(\langle \mathbf{n}, \mathbf{\Delta v}\rangle_{k\ell} U_\ell \otimes \mathbf{x}^{(n)})|_{\mathscr{S}} = n_p \nabla_q \langle\langle qp|k\ell\rangle\rangle (U_\ell \otimes \mathbf{x}^{(n)})|_{\mathscr{S}} \tag{C.4}$$

and the divergence theorem applied to (C.1) over the body's exterior. Applying the anti-symmetry properties (C.3) the integrand can be further transformed as

$$\nabla_p\{\nabla_q \langle\langle qp|k\ell\rangle\rangle (U_\ell \otimes \mathbf{x}^{(n)})\} = \nabla_q\{\langle\langle qp|k\ell\rangle\rangle \nabla_p (U_\ell \otimes \mathbf{x}^{(n)})\} \tag{C.5}$$

enabling the divergence theorem to be applied a second time to convert back to a surface integral. Importantly the contribution from infinity is identically zero since the singular disturbance velocity potential $\Delta\varphi$ can, at infinity, be replaced by its corresponding regular expansion and the argument reversed. The regular form of $\Delta\varphi$ is obtained from its spherical harmonic expansion (Morse and Feshbach, 1953, Part II, p. 1264) by replacing $r^{-(n+1)}$ by $R^{-(2n+1)} r^n$ on the asymptotic sphere S_R with large radius R. In general, therefore, the lift $\mathscr{L}^{(n)}$ can be expressed as

$$\mathscr{L}_k^{(n)} = \int_{\mathscr{S}} n_q \langle\langle qp|k\ell\rangle\rangle \nabla_p (U_\ell \otimes \mathbf{x}^{(n)})\, \mathrm{d}s = \int_{\mathscr{S}} \Delta\varphi\, \mathbf{n} \cdot \mathbf{\Pi} : [\mathbf{\nabla} \otimes (\mathbf{U} \otimes \mathbf{x}^{(n)})]\, \mathrm{d}s. \tag{C.6}$$

By way of example we now evaluate the zero-order ($\mathscr{L}^{(0)}$) and first-order ($\mathscr{L}^{(1)}$) generalised lifts in terms of the generalised Kelvin Impulse $\mathscr{m}^{(n)}$. We shall restrict our analysis to when the ambient velocity has the spatially linear form $U_\ell = U_{\ell|\mathbf{0}} + x_m (\nabla_m U_\ell)|_{\mathbf{0}}$. Writing $\mathbf{x}^{(0)} = 1$ and $\mathbf{x}^{(1)} = x_\alpha$ the square bracketed terms in the integrand of (C.6) for $\mathscr{L}^{(0)}$ and $\mathscr{L}^{(1)}$ are given, respectively, by

$$\nabla_p[U_\ell] = (\nabla_p U_\ell)|_{\mathbf{0}} \tag{C.7a}$$

and

$$\nabla_p[U_\ell x_\alpha] = U_{\ell|\mathbf{0}} \delta_{p\alpha} + (\nabla_m U_\ell)|_{\mathbf{0}} [\delta_{mp} x_\alpha + \delta_{p\alpha} x_m]. \tag{C.7b}$$

Eliminating the fluid velocity divergence $(\nabla_\ell U_\ell)|_{\mathbf{0}}$, because the fluid is incompressible, the integrand of (C.6) is for $\mathscr{L}^{(0)}$

$$n_q \langle\langle qp|k\ell\rangle\rangle \nabla_p [U_\ell] = \Delta\varphi (\nabla_k U_\ell)|_{\mathbf{0}} n_\ell \tag{C.8a}$$

and for $\mathscr{L}^{(1)}$

$$n_q \langle\langle qp|k\ell\rangle\rangle \nabla_p [U_\ell x_\alpha] = \Delta\varphi (n_\ell \delta_{\alpha k} - n_k \delta_{\alpha\ell}) U_{\ell|\mathbf{0}} + \Delta\varphi (\nabla_m U_{\ell})|_{\mathbf{0}} [n_\ell x_\alpha \delta_{mk} + n_\ell x_m \delta_{\alpha k} - n_k x_m \delta_{\alpha\ell}]. \tag{C.8b}$$

In their indexed forms the lifts can now be written

$$\mathscr{L}_k^{(0)} = (\nabla_k U_\ell)|_{\mathbf{0}} \mathscr{m}_\ell^{(0)} \tag{C.9a}$$

and

$$\mathscr{L}_{k\alpha}^{(1)} = (\mathscr{m}_\ell^{(0)} U_{\ell|\mathbf{0}} \delta_{\alpha k} - \mathscr{m}_k^{(0)} U_{\alpha|\mathbf{0}}) + [\mathscr{m}_{\ell\alpha}^{(1)} (\nabla_k U\ell)|_{\mathbf{0}} + \mathscr{m}_{\ell m}^{(1)} (\nabla_m U_\ell)|_{\mathbf{0}} \delta_{\alpha k} - \mathscr{m}_{km}^{(1)} (\nabla_m U_\alpha)|_{\mathbf{0}})] \tag{C.9b}$$

or equivalently, in non-indexed tensor and commutator notation, as

$$\mathscr{L}^{(0)} = \mathscr{m}^{(0)} \cdot (\mathbf{\nabla} \otimes \mathbf{U})^{\mathrm{T}}|_{\mathbf{0}} = (\mathbf{\nabla} \otimes \mathbf{U})|_{\mathbf{0}} \cdot \mathscr{m}^{(0)} \tag{C.10a}$$

and

$$\mathscr{L}^{(1)} = (\mathbf{m}^{(0)} \otimes \mathbf{U}|_{\mathbf{0}}\mathbf{I} - \mathbf{m}^{(0)}\mathbf{U}|_{\mathbf{0}}) + [(\boldsymbol{\nabla} \otimes \mathbf{U})|_{\mathbf{0}}, \mathbf{m}^{(1)}] + (\boldsymbol{\nabla} \otimes \mathbf{U})|_{\mathbf{0}} : \mathbf{m}^{(1)}\mathbf{I}. \tag{C.10b}$$

Appendix D. Kirchhoff expansions of $\mathsf{D}^{(n)}$ and $\mathscr{E}^{(n)}$

The surface momentum flux is given by our (4.7) as

$$\mathsf{D}^{(n)} = \int_{\mathscr{S}} (\boldsymbol{\nabla} \otimes \mathbf{x}^{(n)})\Delta\varphi\,\mathbf{U}^{\mathrm{B}} \cdot \mathbf{n}\,\mathrm{d}s. \tag{D.1}$$

Substituting from our (2.7) and (2.9a) the scalar part of the above integrand can be expressed in terms of the Kirchhoff potentials as

$$\mathbf{n} \cdot \mathbf{U}^{\mathrm{B}}\Delta\varphi = \sum_{m,p=0}^{\infty,\infty} \frac{1}{m!}\frac{1}{p!}(\mathbf{n} \otimes \mathbf{x}^{(m)})|_{\mathscr{S}} \cdot^{(m+1)} (\boldsymbol{\nabla}^{(m)} \otimes \mathbf{U}^{\mathrm{B}})|_{\mathbf{0}}\boldsymbol{\psi}^{(p)}|_{\mathscr{S}} \cdot^{(p+1)} (\boldsymbol{\nabla}^{(p)} \otimes \boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}. \tag{D.2}$$

By differentiating-out the term $\boldsymbol{\nabla} \otimes \mathbf{x}^{(n)}$, and combining with (D.2), the integrand of (D.1) can be expanded as a sum over the tensors $\mathbf{x}^{(n-1)} \otimes (\mathbf{n} \otimes \mathbf{x}^{(m)}) \otimes \boldsymbol{\psi}^{(p)} \cdot \mathsf{D}^{(n)}$, therefore, has the following general expansion in the coefficients of the tensor $\mathbf{m}^{(n+m-1,p)}$

$$\mathsf{D}^{(n)} = \sum_{n,m,p,*} \frac{1}{m!}\frac{1}{p!}\mathbf{m}_*^{(n+m-1,p)} \cdot^{(m+p+2)} (\boldsymbol{\nabla}^{(m)} \otimes \mathbf{U}^{\mathrm{B}})|_{\mathbf{0}} \otimes (\boldsymbol{\nabla}^{(p)} \otimes \boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}}. \tag{D.3}$$

Here $*$ denotes the appropriate summation over the indices of the $(n+m+p+1)$th-order tensor $\mathbf{m}^{(n+m-1,p)}$.

The surface kinetic energy is given by our (4.9) and (4.5c) as

$$\mathscr{E}^{(n)} = -\int_{\mathscr{S}} \left\{ \frac{1}{2}\boldsymbol{\Delta}\mathbf{v} \cdot \boldsymbol{\Delta}\mathbf{v}\,\mathbf{I} - \boldsymbol{\Delta}\mathbf{v} \otimes \boldsymbol{\Delta}\mathbf{v} \right\} \otimes \mathbf{x}^{(n)}\,\mathrm{d}s. \tag{D.4}$$

The scalar and tensor products of the disturbance velocity in the integrand of (D.4) can be expanded by substituting from our (2.9a) and making the appropriate tensor rearrangements to give

$$\mathscr{E}^{(n)} = \sum_{m,p=0}^{\infty,\infty} \frac{1}{m!}\frac{1}{p!}(\boldsymbol{\nabla}^{(m)} \otimes \boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}} \otimes (\boldsymbol{\nabla}^{(p)} \otimes \boldsymbol{\delta}\mathbf{U})|_{\mathbf{0}} \cdot^{(m+p+2)} \mathscr{E}^{(n,m,p)} \tag{D.5a}$$

where $\mathscr{E}^{(n,m,p)}$ is our added Kirchhoff energy tensor

$$\mathscr{E}^{(n,m,p)} = \int_{\mathscr{S}} \left(\left\{ \frac{1}{2}(\boldsymbol{\psi}^{(m)} \otimes \boldsymbol{\nabla}) \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{\psi}^{(p)})\mathbf{I} - (\boldsymbol{\nabla} \otimes \boldsymbol{\psi}^{(m)}) \otimes (\boldsymbol{\psi}^{(p)} \otimes \boldsymbol{\nabla}) \right\} \cdot \mathbf{n} \right) \otimes \mathbf{x}^{(n)}\,\mathrm{d}s. \tag{D.5b}$$

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